On Characterization of Norm Attaining Operators in Fréchet Spaces

Vashti Bernice [a*], Amos Wanjara [a] and Michael Ojiema [b]

[a]*Department of Mathematics and Statistics, Kaimosi Friends University, P.O. Box 385-50309, Kaimosi, Kenya.*

[b]*Department of Mathematics, Masinde Muliro University of Science and Technology, P.O. Box 190-50100, Kakamega, Kenya.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Original Research Article

Abstract

In this paper, the norm attaining operators in Fréchet spaces are considered. These operators are characterized based on their density, normality, linearity and compactness. It is shown that the image is dense for a normal and injective operator in a Fréchet space, as well as its inverse given that the operator is self-adjoint. A norm attaining operator in a Fréchet space is also shown to be normal if its adjoint also attains its norm in the Fréchet space, and the condition under which the norm attainability and the normality of an operator in a Fréchet space coincides is given. Furthermore, a norm attaining operator between Fréchet spaces is linear and bounded as well as its inverse. If a norm attaining, normal and dense operator is of finite rank, then it is compact. The study of norm attaining operators is applicable in algorithm concentration as seen in describing sphere packing.

Keywords: Norm attaining operator; Fréchet space; Characterization.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16.

_____

*Corresponding author: E-mail: vashtibernice@gmail.com;

1   Introduction

In honor of Maurice Fréchet, special topological vector spaces are known as Fréchet spaces. The idea of Fréchet spaces generalizes the concepts of Hilbert spaces (complete inner product (or pre-Hilbert) spaces) and Banach spaces (normed vector spaces that are complete with regard to the metric produced by the norm), both of which are crucial concepts in operator theory [1],[2]. Oftenly, investigations on norm attaining operators are done in Banach spaces. However, it is usually more natural and convenient to put considerations in appropriate Fréchet function space instead of Banach function space since a Fréchet space has additional properties including local convexity and metrizability.

By supplying the necessary conditions for the expansion of a topological space by the sequential closure operator to be a Fréchet space as well as a sufficient condition for a topological space to simply expand to be Fréchet, Woo Chorl [3] enriched sequential convergence structures and characterized Fréchet spaces. Vogt [4] examined the relationship between Fréchet spaces and Banach spaces. The concept of a Banach space is well-known [5]. Therefore, this study focused on Fréchet spaces as ambient spaces of every Banach space hence generalizing Banach spaces.

Banach spaces provide for a variety of structures, including topological, geometrical, and algebraic ones. According to [6], these might be thought of as topological, metric, or linear spaces. The norm topology, which is caused by the metric produced by the norm, is one of the natural topologies that are admissible in Fréchet spaces. The weak topology is the weakest topology and shares the same continuous linear functions as the norm topology [cf. [7]].

Suppose that B is a bounded closed convex subset of a Banach space Y with $y \in B$, where $ry(B) < d(B)$ with

$$\begin{aligned} ry(B) &= sup\{\|y - z\| : z \in B\} \, \forall y, z \in B \\ d(B) &= sup\{\|y - z\| : y, z \in B\} \end{aligned}$$

Consequently, Y is considered to have a normal structure. In other words, Banach space B has normal structure if there is a point p in each closed bounded convex set K in B that has at least one point such that $sup\{\|p - x\| : x \in K\}$ is less than the diameter of K [8].

Fréchet spaces admit several operators including sequential closure operators and norm attaining operators. However, the study of norm attaining operators provides certain familiarity with the geometric aspects of the Radon-Nikodym property, like the dentability. As a result, characterization of norm attaining operators has been done, primarily on the denseness property.

According to Bishop-Phelps theorem [9], the collection of norm-attaining functionals on a Banach space is dense in the dual. However, in the complex situation, it is possible for a closed, convex, and bounded subset of a Banach space, C, to exist such that the set of functionals whose maximum modulus is achieved on C is not dense in the dual [10].

Some isometric conditions on X for which the set of norm attaining operators from X to Y are dense in the space of all operators between these Banach spaces were provided by Lindestrauss [11]. Specifically, consider X and Y as two Banach spaces. If $NA(X, Y)$ is dense in $L(X, Y)$ for any Banach space Y, then a Banach space X is said to have *propertyA*. Additionally, if $NA(X, Y)$ is dense in $L(X, Y)$ for any Banach space X, then a Banach space Y is said to have *propertyB*. According to Lindestrauss [[11], proposition 4], if $B(C_0, X)$ contains a non-compact operator and X is strictly convex, then X does not possess *propertyB*. The research continued, in regard to the general question posed by Bishop and Phelps [12]. It was demonstrated that for some space Y, the

norm attaining operators in $B\left(L^1\left[0,1\right],Y\right)$ are not dense because there are no extreme points in the closed unit ball of $L^1\left[0,1\right]$. However, Johnson [13] confirmed the Radon-Nikodym property's close link to extreme point structure in relation to the denseness of norm attaining operators for strictly convex Banach spaces Y. The study also attempted to address the unresolved issue of the density of the norm attaining operators in $B\left(L^1\left[0,1\right],Y\right)$ in the Banach space Y.

It has been demonstrated that an operator of a particular class can be approximated by an operator of the same class that attains a norm. According to Johnson [14], similar conclusions for the idea of norm attaining operators taking into account an operator between compact Hausdorff spaces have also been developed.

Venku [15] provided necessary and sufficient conditions for a limited operator formed between complex Hilbert spaces to be completely norm attaining. The structure of such operators in the case of self-adjoint and normal operators was also examined separately.

Okelo [16] gave the characterizations of both the power and non-power operators considering the Banach space setting. They considered norm-attainability for inner derivation, generalized derivations and general elementary operators [17]. However, the generalizations still remained open in Fréchet spaces.

From the foregoing, having such generalizations in Fréchet spaces remained of interest. Additionally, it was intriguing to characterize the density of norm attaining operators in norm attaining operator classes other than $L^1\left[0,1\right]$. Are there any additional characteristics for characterizing norm attaining operators besides density, too? This unanswered question served as a clear impetus for the investigation on the characterization of norm attaining operators. The following fundamental ideas were crucial for conducting the research.

1.1 Basic Concepts

Definition 1.1 ([18], Definition 1.10.2)**.** A *vector space* over a scalar field $\mathbb{F}$ is a set V that satisfies vector addition and scalar multiplication. A vector space is also called a linear space. The elements of a vector space are called vectors. The trivial vector is $V = 0$.

Definition 1.2 ([19], Definition 3.1.1)**.** Assume that X is a vector space with all vectors x, y contained within it, and that $\mathbb{F}$ is a scalar field with all scalars c contained within it. A function $\|\cdot\| : X \rightarrow \Re$ satisfying

1. non-negativity: $0 \leq \|x\| < \infty$

2. homogeneity: $\|cx\| = |c|\|x\|$ and

3. triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

is said to be a seminorm on X and is a norm if it also satisfies
   4. uniqueness (zero axiom) : $\|x\| = 0$ iff $x = 0$.

Definition 1.3 ([20], Definition 2.0)**.** Assume that V is a vector space with a defined norm. Then, V is referred to as a normed space, also known as a Pre-Banach space. A metric $d\left(v,w\right) = \|v - w\|$ is inherently linked to a normed space V and it is often referred to as the translation invariant. The normed space V is referred to as a Banach space if it is complete with the corresponding metric.

Definition 1.4 ([21], Definition 1.4)**.** Given a complex vector space V, a complex-valued function $\langle,\rangle : V \times V \rightarrow C$ of two variables on V is an inner product if

1. $\langle x, y \rangle = \langle \bar{y}, x \rangle$

2. $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$

3. $\langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle$

4. $\langle x, x \rangle \geq 0$

5. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

6. $\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$

V equipped with such a $\langle , \rangle$ is a pre-Hilbert space. And a complete pre-Hilbert space, say H, that is,

$$\lim_{n,m \to \infty} \langle x_n - x_m, x_n - x_m \rangle = 0 \forall x_n \in H \forall n \in \mathbb{N}, m \in \mathbb{R}$$
$$\Rightarrow \exists x \in H : \lim_{n \to \infty} \langle x - x_n, x - x_n \rangle = 0$$

is a Hilbert space.

Definition 1.5 ([22], Definition 1.1). Let $d : Y \times X \to E'$ be a map that meets the conditions listed below:

1. $d(x, y) \geq 0$ for each pair x, y

2. $d(x, y) = 0$ iff $x = y$

3. $d(x, y) = d(y, x) \forall x, y$ (symmetry)

4. $d(x, z) \leq d(x, y) + d(y, z) \forall x, y, z$ (triangle inequality)

The map $d : Y \times X \to E'$ is referred to as a metric (or distance function) on the set Y. The distance between x and y, denoted by $d(x, y)$ is known as the metric and the set Y endowed with d, denoted by (Y, d), is known as the metric space.

Definition 1.6 ([22], Definition 2.2). Let (Y, d) be a metric space. Given a topology $\tau(d) \in Y$ such that $\tau(d)$ is determined by d, the topological space $(Y, \tau(d))$ is referred to as a metrizable space if $\tau(d) \in Y$ is such that $\tau(d)$ is determined by d.

Definition 1.7 ([22], Definition 2.3). $(Y, \tau(d))$ is said to be completely metrizable if the metric space (Y, d) such that d induces τ is complete.

Definition 1.8 ([21], Definition 2.5). A local basis (0-neighborhood basis) at $u \in U$ is made up of open balls centered at $U, \{v \in U : d(u, v) < r\}$.

Definition 1.9 ([4], Definition 1.3). If a space has a basis of absolutely convex neighborhoods of zero, it is said to be locally convex.

Definition 1.10 ([4],[21], Definition 1.5). A complete topological vector space with absolutely convex neighborhoods of zero is known as a Fréchet space.

Definition 1.11 ([15], Definition 1.9). Let H_1 and H_2 be vector spaces. A linear operator $T : H_1 \to H_2$ for all $x \in H_1$, is said to be bounded if there exists an integer $c > 0$ such that $\|Tx\| \leq c\|x\|$.

Definition 1.12 ([14], Definition 2.2). Let x_0 be a unit vector in X. For Banach spaces X and Y, if a bounded linear operator $T : X \to Y$ exhibits the property
$\|Tx_0\| = \|T\| = sup\|Tx\| : x \in X and \|x\| \leq 1$, it is referred to as a norm attaining operator.

Definition 1.13 ([23], Definition 2.5). A topological space denoted by (X, τ) is a non-empty set X together with the collection τ of subsets of X(referred to as open sets) that satisfies the following conditions:

1. The empty set Φ and the whole space X are members of τ(i.e, are open sets).

2. The union of any collection of open sets is itself an open set.

3. The intersection of any finite collection of open sets is itself an open set.

Definition 1.14 ([13], Definition 2.6). Let Y be a Banach space that is strictly convex. Then, in the space of all linear operators from $L^1[0, 1]$, the norm attaining operators mapping $L^1[0, 1]$ to Y are dense if and only if Y possesses the Radon-Nikodym property.

2 Research Methods

2.1 Bishop-Phelps Property

Let C be a convex, closed, and bounded subset of X and X be a Banach space. According to [9], C is said to satisfy the Bishop-Phelps property if for any Banach space Y, the set of operators $T \in L(X, Y)$ such that the function $x \to \|Tx\|$ attains its maximum in C is dense in $L(X, Y)$.

A Banach space X clearly possesses *property A* if and only if its unit ball B_X possesses Bishop-Phelps property. If every convex, closed, and bounded subset of a Banach space X possesses the Bishop-Phelps property (BPp), then X is said to have this property.

In order to characterize the density of norm attaining operators with respect to the topology of the norm, Bishop-Phelps property was helpful.

2.2 Radon-Nikodym Property

Let $G : \Sigma \to X$ be a μ-continuous vector measure with $g \in L^1(\mu, X)$ such that $G(E) = R E g d_\mu$ for all $E \in \Sigma$. Then, X has the *Radon-Nikodym property* according to (Ω, Σ, μ) [24].

A subset is said to have the Radon-Nikodym property if every non-empty subset of a Banach space X is dentable, or if every subset of C has slices of any diameter. According to Phelps' results, the Radon-Nikodym property, which is expressed in terms of measure theory, holds for any non-empty bounded subset of a Banach space X [19].

Recall: The norm attaining operators mapping $L^1[0, 1]$ to Y are *dense* in the space of all linear operators from $L^1[0, 1]$ to Y if and only if Y possesses the Radon-Nikodym property and is strictly convex [13].

Lemma 2.1 ([24], Theorem 3.2.3). *(The Radon-Nikodym Theorem) Let λ be a real measure on M and μ be a positive measure with a σ-finite range. There is only one g in $L^1(\mu)$ such that $d\lambda_a = g d_\mu$. $g \geq 0 a.e [\mu]$ if λ is a finite positive measure.*

Proof. Suppose $g_k \in L^1(\mu), k = 1, 2...$ and $d\lambda_a = g_1 d_\mu = g_2 d_\mu$. Then for any arbitrary h, we have $h d_\mu = 0$ where $h = g_1 - g_2$. But then $\int_{h \geq 0} h d_\mu = 0$ and it follows that $h \leq 0 a.e [\mu]$. It is similarly proven that $h \geq 0 a.e [\mu]$. Thus $h = 0$ in $L^{1\infty}(\mu)$, that is, $g_1 = g_2$ in $L^1(\mu)$. □

Lemma 2.2 ([19], Theorem 5). *Let $C \subset X$ be a Radon-Nikodym set that is non-empty, bounded, closed, and absolutely convex. Then, for any Banach space Y, a G_δ-dense set in $L(X, Y)$ exists in the subset of operators $T \in L(X, Y)$ such that $sup_{x \in C} \|Tx\|$ is achieved.*

In order to characterize the operators that attain their norms in consideration of their densities with regard to each finite measure space, the Radon-Nikodym property was useful.

3   Results and Discussion

3.1   Density of Norm Attaining Operators in Fréchet Spaces

Proposition 3.1. *A bounded linear operator T attain its maximum if and only if the unit ball possesses the Bishop-Phelps property.*

Proof. The proof is given for the sufficiency part. Take into consideration a Fréchet space E's convex, closed, and bounded subset B. From the hypothesis, the unit ball B has Bishop-Phelps property. It naturally follows that the set of operators $T \in L(E, F)$ is dense in $L(E, F)$ for the topology norm for every space Fréchet F. This suggests that the unit ball B is where the function $f \to \|Tf\| : f \in E$ attains its maximum. □

Theorem 3.1. *Let $T \in B(F)$ for every Fréchet space F, be normal and injective. Then $Im(T)$ is dense.*

Proof. For any operator $T \in B(F)$, $kerT^* = (ImT)^\perp$. Since T^* is restricted to the image of T, T^* is injective and therefore, $kerT = kerT^*T$. But, T is also normal. This implies that

$$
\begin{aligned}
kerT &= kerT^*T = kerTT^* \\
&= kerT^* = (ImT)^\perp
\end{aligned}
$$

It then follows that T is injective if and only if the orthogonal complement of the image is trivial, which implies that T has dense image. □

Lemma 3.2. *Assume that operator T is self-adjoint. If $T \in NA(E, F)$ and $NA(E, F)$ is dense in $L(E, F)$, then $T^* \in NA(E, F)$ for any Fréchet space E and Fréchet space F.*

Proof. The proof immediately follows from the fact that a densely defined linear operator that maps a Fréchet space onto itself and is invariant under the unary operation of taking the adjoint is self-adjoint. □

Theorem 3.3. *A Fréchet space for some $L \in Sc[X]$ is defined as (X, c_L). Give an example of $c_L : P(X) \to P(X)$. It is claimed that c_L is norm attaining if there is a unit vector $x \in X$ such that $\|c_L x\| = \|c_L\|$. The set of operators in $P(X)$ is hence dense in $P(X)$.*

The following remark will be necessary for the proof.

Remark 3.1. Every sequentially continuous linear operator between topological vector spaces is a bounded operator.

Proof. The remark above indicates that c_L is bounded. Now, if X is a normed vector space (a special type of a Fréchet space), then c_L is bounded by definition if and only if some $M > 0$ exists such that for any $x \in X, \|c_L x\|_{P(X)} \leq M\|x\|_{P(X)}$. This implies that the operator norm M exists for c_L. That is to say, $\|c_L\| = M$. Additionally, it follows that $x \in X$ is a unit vector. In other words, $\|x\| = 1$. Therefore,

$$
\begin{aligned}
\|c_L x\|_{P(X)}\| &\leq M\|x\|_{P(X)} \\
&= \|c_L\|\|x\|_{P(X)} \\
&= \|c_L\|_{P(X)}
\end{aligned}
$$

Because of this, c_L is attaining the norm. The set of norm attaining operators from $P(X)$ is hence dense in $L[P(X), P(X)]$ if $P(X)$ possesses the Radon-Nikodym property. □

In addition to density, the normality, linearity, and compactness properties are used to characterize norm attaining operators in Fréchet spaces.

3.2 Normality of Norm Attaining Operators in Fréchet Spaces

Proposition 3.2. *Assume that F is a Fréchet space and that $T, T^* \in AN(F)$. Consequently, $T \in B(F)$ is normal.*

Proof. Assume that $T \in B(F)$ and $T^* \in AN(F)$ are both normal. It must be demonstrated that $T \in AN(F)$. The fact that $T^* \in AN(F)$ if and only if $TT^* \in AN(F)$ is well known. Additionally, the fact that $T \in B(F)$ is normal implies that $T^*T = TT^*$. Consequently, $T^*T \in AN(F)$ which also suggests that $T \in AN(F)$. □

Theorem 3.4. *For a self-adjoint operator T, the norm attainability and the normality of T coincides.*

Proof. Given that T is a self-adjoint operator, it must be demonstrated that T is also normal if T is norm attaining. Let E be a Fréchet space and T be a norm-attaining operator so that $T \in B(E)$. If E possesses the Radon-Nikodym property and is strictly convex, then $T \in NA(E)$ is dense in E. It follows that T is self-adjoint. Therefore, according to Proposition 3.2, if $T \in NA(E)$, then also $T^* \in NA(E)$, indicating that $TT^* = T^*T \in NA(E)$ (Lemma 3.2). Therefore, T is normal. □

3.3 Linearity of Norm Attaining Operators in Fréchet Spaces

Theorem 3.5. *A norm-attaining operator between the Fréchet spaces E and F is defined as $T \in NA(E, F)$. As a result, T is linear and bounded.*

Proof. Let T attain its norm. Then a unit vector f_0 exists in E such that $\|Tf_0\| = \|T\|$. The rewritten version of this is $\|Tf_0\| = \|T\|\|f_0\| = \|T\|.1 = 1\|T\|$. It follows that T is bounded because the coefficient of $\|T\|$ is a real number. Now that T is bounded and norm-attaining, its linearity follows naturally.

Conversely, assume that T is bounded and linear. For any real (or complex) number $\lambda > 0$, $\|T(f_0)\| \leq \lambda \|f_0\|, \forall f_0 \in E$. But, $\|f_0\| = 1$. Consequently, $\|T(f_0)\| \leq \lambda$. According to the equality, λ must equal $sup\|T(f_0)\|$. Finally, it can be seen that given the unit vector f_0 in E, $\|Tf_0\| \leq \lambda = \|T\| = sup\|T\|$ and so T attains its norm. □

Theorem 3.6. *Assume that $T : E \to F$ is onto and one-to-one such that T^{-1} exists. If and only if T is linear, then T^{-1} is linear.*

Proof. Let $e, f \in E$ and $g, h \in F$. From the hypothesis, T is bijective. Therefore, $T(f) = g$ and $T(e) = h$. $T^{-1} : F \to E$ is then defined by

$$T^{-1}(g) = f \quad and \quad T^{-1}(h) = e \tag{3.1}$$

so that $TT^{-1} = I, T^{-1}T = I$. Suppose that T is linear. Then for all $e, f \in E$,

$$T(e + f) = T(e) + T(f) \quad and, \quad T(cf) = cT(f) \forall f \in E \quad and \quad c \in K. \tag{3.2}$$

Substituting 3.1 into 3.2,

$$T(T^{-1}(h) + T^{-1}(g)) = T(T^{-1}(h)) + T(T^{-1}(g))$$
$$\Rightarrow TT^{-1}(h + g) = TT^{-1}(h) + TT^{-1}(g). \tag{3.3}$$

But $TT^{-1} = I = T^{-1}T$. Therefore, 3.3 can be re-written as

$$T^{-1}T(h + g) = T^{-1}T(h) + T^{-1}T(g)$$
$$\Rightarrow T^{-1}(T(h) + T(g)) = T^{-1}(T(h)) + T^{-1}(T(g)). \tag{3.4}$$

Again, $T(cf) = cT(f)$ will be such that

$$
\begin{aligned}
Tc(T^{-1}(g)) &= cT(T^{-1}(g)) \\
&= cTT^{-1}(g) \\
&= cT^{-1}T(g)(since TT^{-1} = I = T^{-1}T) \\
\Rightarrow T^{-1}c(T(g)) &= cT^{-1}(T(g)).
\end{aligned}
$$

Hence, T^{-1} is linear. $\qquad\square$

3.4   Compactness of Norm Attaining Operators in Fréchet Spaces

The following lemma is a useful tool for the proof of the proposition that follows. It is stated without its proof.

Lemma 3.7. *Every closed, bounded subset of $\mathbb{F}^k$ is compact.*

Proposition 3.3. *Let $T \in B(E, F)$ be a norm attaining operator of finite rank for Fréchet spaces E and F. T is then compact.*

Proof. From the hypothesis, T is of finite rank. Therefore, there exists a finite dimensional space, say $G = ImT$. Suppose that a sequence e_n is bounded in E. It follows that the sequence Te_n is bounded in G and by Lemma 3.7, there must be a convergent subsequence in this sequence. T is hence compact. $\qquad\square$

Theorem 3.8. *Let $T \in L(E, F)$ be a compact operator for Fréchet spaces E and F. T then attains its norm if $[kerT]^{\perp} \subset NA(E)$.*

Proof. According to James' theorem, $E/kerT$ must be reflexive since
$[kerT]^{\perp} \subset NA(E)$. Now, T factors through $E/kerT$. In other words, a mapping
$S : E/kerT \to ImT \in F$ exists such that $E/kerT \cong ImT$, and $T = S \circ \delta$. It is evident that
$\|S\| = \|T\|$. From the hypothesis, T is compact. As a result, the sequence Te_n is bounded in F and contains a convergent subsequence for any bounded sequence $e_n \in E$. Due to the fact that T factors via $E/kerT$, e_n will converge through $E/kerT$ in such a way that Se_{n_r} is bounded in $ImT, \forall e_{n_r} \in E/kerT$. Since Se_{n_r} will have a convergent subsequence according to Lemma 3.7, S is implied to be compact whenever T is. Then, according to Proposition 3.3, S attains its norm. As a result, the adjoint S^* also attains its norm. In other words, there is a $f^* \in S_{F^*}$ such that $\|S^*f^*\| = \|T\|$. The functional $e^* = T^*f^* = [\delta^*S^*](f^*) \in E^*$ now disappears on $kerT$, meaning that $e^* = T^*f^* \Rightarrow Te^* = f^*$. But, $kerT := \{e^* : Te^* = 0 \neq f^*\}$. So, $e^* \in [kerT]^{\perp} \subset NA(E)$. This suggests the existence of $e \in S_E$ in such a way that

$$
\begin{aligned}
|e^*(e)| &= \|e^*\| \\
&= \|[\delta^*S^*](f^*)\| \\
&= \|\delta^*(S^*f^*)\| \\
&= \|S\|
\end{aligned}
$$

where δ^* embeds isometrically. Consequently, $\|T\| = |[T^*f^*](e)| = |f^*(Te)|$ and so
$\|Te\| = \|T\|$. $\qquad\square$

Lemma 3.9. *Let $T \in B(E, F)$. The dimension of E or the dimension of F must be finite for T to be compact.*

Proof. If the dimension of E is finite, then the rank of T is finite because $r(T) \leq dim(E)$. However, it is obvious that the dimension of $ImT \subset F$ must be finite if the dimension of F is finite. The conclusion so comes from the demonstration of Proposition 3.3 in either scenario. $\qquad\square$

The following proposition is necessary for the proof of the theorem that follows. It is stated without proof.

Proposition 3.4. *Given a normed space E and a sequence of bounded, finite rank operators T_n that converge to $T \in B(E, F)$, T is compact for the Fréchet space F.*

Theorem 3.10. *If $T \in B(F)$ and F is a Fréchet space, then the compactness of T^* implies the compactness of T, and vice versa.*

Proof. Assume that T is compact. Then, a set of finite operators T_n exists such that $\|T_n - T\| \to 0$. However, $r(T) = r(T^*)$. As a result, every operator T_n^* possess a finite rank, and since $\|T^*\| = \|T\|$, $\|T_n^* - T^*\| = \|T_n - T\| \to 0$, respectively. It follows that T^* is compact from Proposition 3.4. Conversely, if T^* is compact, then T is compact because of this result and from the fact that $(T^*)^* = T$. □

The implications established thus far are now addressed, and the prospect of undoing them is discussed.

Theorem 3.11. *Let $T \in B(E, F)$ be a bounded linear operator for Fréchet spaces E and F. Consider the following properties:*

1. *T is norm attaining.*
2. *If and only if T is self adjoint and $NA(E, F)$ is dense in $L(E, F)$, T^* is norm attaining.*
3. *T is normal.*
4. *ImT is dense.*
5. *$[kerT]^\perp \subset NA(E)$.*
6. *T is compact.*
7. *T^* is compact.*
8. *$Dim(E)$ or $Dim(F)$ is finite.*

The implications are as follows:

$$1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftarrow 4 \Leftrightarrow 5 \Leftarrow 6 \Leftrightarrow 7 \Leftarrow 8$$

Proof. From Lemma 3.2, it is immediate that $1 \Rightarrow 2$, and also that $2 \Rightarrow 1$. Proposition 3.2 and Theorem 3.4 give the implication $2 \Leftrightarrow 3$. The implication $4 \Rightarrow 3$ is immediate from Theorem 3.1, however, its converse need not be true. Next, it is only shown that $4 \Rightarrow 5$ since its reverse implication is obvious. Now, by 4, $ImT \in NA(E, F)$ implying that its orthogonal complement is trivial, by Theorem 3.1. But,

$$\begin{aligned}(ImT)^\perp &= kerT \\ \Rightarrow kerT &\nsubseteq NA(E, F) \\ \Rightarrow [kerT]^\perp &\subset NA(E, F).\end{aligned}$$

Lastly, the implications $5 \Leftarrow 6 \Leftrightarrow 7 \Leftarrow 8$ follow immediately from Theorem 3.8, Theorem 3.10 and Lemma 3.9 respectively. However, the reverse implications $5 \Rightarrow 6$ and $7 \Rightarrow 8$ need not be true. □

The following observations are made from Theorem 3.11:
Observations

- All the other properties imply 1;
- All other properties, including 1 imply 3;
- $1 \Rightarrow 3$ and $3 \Rightarrow 1$. Therefore, the norm attainability and normality of T coincides given that T is self adjoint (which is Theorem 3.4).

4 Conclusions

a In the subsection (3.1), the density of norm attaining operators has been discussed whhereby, it has been shown that if an operator is normal and injective in a Fréchet space, then its image is dense. Also, it has been shown that the adjoint of a dense operator in a Fréchet space is also dense provided that the operator is self-adjoint.

b In the subsection (3.2), it has been shown that a norm attaining operator in a Fréchet space is normal if its adjoint also attains its norm in the Fréchet space. Also, the condition under which the norm attainability and the normality of an operator in a Fréchet space coincides has been shown.

c In the subsection (3.3), it has been shown that a norm attaining operator between Fréchet spaces is linear and bounded as well as its inverse.

d In the subsection (3.4), the compactness of an operator of finite rank has been discussed.

However, the characterization of norm attaining operators in Fréchet spaces is not limited to the density, normality, linearity and compactness properties. One can therefore, consider nuclearity property and check whether there is a relationship between nuclearity of norm attaining operators in Fréchet spaces and in other spaces.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Alnoor OF. A note on norm attaining operators. IOSR Journal of Mathematics (IOSR-JM). 2018;14:78-80.
e-ISSN: 2278-5728, p-ISSN: 2319-765X.

[2] Huff RE. Dentability and the Radon-Nikodym Property. Duke Math. J. 1974;41:11-114.

[3] Woo C. Notes on Fréchet spaces. Internat. J. Math. and Math. Sci. S 0161-1712(99)22659-2, Electronic Publishing House. 1999;22(3):659-665.

[4] Vogt D. Lectures on Fréchet Spaces; Definition and basic properties. Bergische Universität Wuppertal, Sommersemester; 2000.

[5] Martin M. Norm-attaining compact operators. J. Funct. Anal. 2014;267:1585-1592.

[6] Wanjara A. Rotundity of Norms in Fréchet Spaces; 2019.
Available:http://ir.sjooust.ac.ke:8080/xmlui/handle/123456789/8797

[7] Ondiej K. Compact spaces and their applications in Banach space theory. Charles University in Prague, Faculty of Mathematics and Physics, Sokolovska'. 1999;83:186 75 Prague, Czech Republic.

[8]   Lim T. Fixed points of isometries on weakly compact convex sets. Anal. Appl. J. Math. 2003;282(1):1-7.
DOI:10.1016/S0022-24X(03)00398-6

[9]   Lomonosov V. The bishop-phelps theorem fails for uniform non-self adjoint dual operator algebras. J. Funct. Anal. 2001;185:214-219.

[10] Acosta MD. Denseness of norm attaining mappings. Rev. R. Acad. Cien. Serie A. Mat. 2006;100:9-30.
Available:http://www.rac.es/ficheros/doc/00231.pdf

[11] Lindestrauss J. On operators which attain their norm. Israel J. Math. 1963;1:139-148.

[12] Bishop E, Phelps RR. A Proof that every Banach Space is Subreflexive. Bull. Amer. Math. Soc. 1961;67:9798.

[13] Uhl J. Norm attaining operators on $L_1[0, 1]$ and the Radon-Nikodym property. Pacific J. Math. 1976;63:293-300.

[14] Johnson J, Wolfe J. Norm attaining operators. Studia Math. 1979;65:7-19.

[15] Venku N, Ramesh G. On absolutely norm attaining operators. Proc. Indian Acad. Sci (Math. Sci.). 2019;129:54.

[16] Okelo NB. On norm-attainable operators in Banach spaces. Hindawi, Journal of Function Spaces; 2020. Article ID 8926214, 1-6.
Available:http://doi.org./10.1155/2020/8926214

[17] Okelo NB. The norm attainability of some elementary operators. Applied Mathematics E-Notes. 2013;13:1-7.
ISSN: 1607-2510.

[18] Heil C. Metrics, Norms, Inner Products and Operator Theory (Applied and Numerical Harmonic Analysis). Springer International Publishing AG, part of Springer Nature; 2018.

[19] Robert A. Notes on Vector and Matrix Norms, Texas; 2014.

[20] Garret P. Banach and Fréchet Spaces of Functions; 2014.
Available:http://www.math.umn.edu./ garret/m/fun/notes-2012-13/02-spaces-fcns.pdf

[21] Andreas K. Fréchet Spaces. Vienna, 442502, SS 2016, Mo.+Do. $10^{05} - 11^{15}$ SR9; 2016.

[22] Dugundji J. Topology. Allyn and Bacon Series in Advance Mathematics, Inc., Boston; 1966.

[23] Mogotu PO, et al. On denseness of similarity orbits of norm-attainable operators. Int. J. Appl Math and Mech. 2019;6(4):14-21.
ISSN: 2347-2529.

[24] Angela CC. Norm-attaining operators; results on norm-attaining operators. Master Thesis, Universidad Autonoma de Madrid; 2015.