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Matrices of the Zero Divisor Graphs of Classes of 3-Radical Zero Completely Primary Finite Rings

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ABSTRACT

The study of finite completely primary rings through the zero divisor graphs, the unit groups and their associated matrices, and the automorphism groups have attracted much attention in the recent past. For the Galois ring R' and the 2-radical zero finite rings, the mentioned algebraic structures are well understood. Studies on the 3-radical zero finite rings have also been done for the unit groups and the zero divisor graphs $\Gamma(R)$. However, the characterization of the matrices associated with these graphs has not been exhausted. It is well known that proper understanding of the classification of zero divisor graphs with diameter 2 and girth 3 can provide insights into the structure of commutative rings and their zero divisors. In this study, we consider a class of 3-radical zero completely primary finite rings whose diameter and girth are 2 and 3 respectively. We enhance the understanding of the structure of such rings by investigating their Adjacency, Laplacian and Distance matrices.

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Keywords: 3-radical zero finite rings, Completely Primary Finite Rings and Zero divisor graph.

1 Introduction

Throughout the paper, R represents 3-radical zero completely primary finite ring, $\Gamma(R)$ is the zero divisor graph of R and $R' = GR(p^{kr}, p^k)$ denotes the Galois ring of order p^{kr} and characteristic p^k for some positive integers k, r. $Z(R)$ and $J(R)$ are subsets of zero divisors and Jacobson radical respectively. R^* denotes the unit group of R. Unless stated otherwise, other notations are standard.

A finite unital ring R is called completely primary if the subset of its zero divisors forms a maximal ideal[16]. These classes of finite rings have been used in the classification of all other classes of finite rings with identity $1 \neq 0$. The Galois rings, the maximal submodules of finite rings are the trivial classes of completely primary finite rings vital in the classification of rings of idealization. They have been studied by a number of authors for the structures of their unit groups and zero divisor graphs up to isomorphism, (see for example[3, 9]) among others. The 2-radical zero completely primary finite rings have also been classified using the unit groups, the zero divisor graphs, the automorphisms of the graphs and the adjacency and incidence matrices, [3, 10, 14]. In this paper, we investigate the matrices of zero divisor graphs of a class of finite rings with the unique maximal ideal $Z(R)$ such that for any fixed s, $t = \frac{s(s+1)}{2}$ where s and t are the dimensions of R'-modules U and V respectively. This is an extension of the studies in [3, 4, 5]. Indeed, in [5], a classification of such classes of rings R with Jacobson radical $J(R)$ satisfying the condition $(J(R))^3 = (0)$ and $(J(R))^2 \neq (0)$ was considered where the enumeration problem was solved for all characteristics of R. Further, a case of the ring whose characteristic is p^k, $1 \leq k \leq 3$ was considered in [4] by determining the structures of R^*, for $R = R' \oplus U \oplus V$ where U and V are R'-modules generated by s and t generators where $t = \frac{s(s+1)}{2}$ for any fixed s. Another classification attempt was done in [3] where the automorphisms of R were obtained when characteristic of $R = p$. From this study, a characterization was done for the cases in which $\mathbb{F}$ is the Galois field $GF(p^r)$ and $1 \leq t \leq s^2$ for a fixed s, t-dimensional $\mathbb{F}$ spaces U, V respectively and $(a_{ij}^k) \in \mathbb{M}_{s\times s}(\mathbb{F})$ are t linearly independent matrices. The studies mentioned however, did not provide a classification on the structures of zero divisors and by extension, their graphs and matrices associated with the graphs.

The concept of zero divisor graphs has also attracted active research since its inception by Beck as demonstrated in [2]. Other researchers broadened the study based on their choices of zero divisors from R or annihilator ideals of R as evident in [1] and [13]. Other ring characterization via the automorphism of their zero divisor graphs have also been done by some researchers. In [9, 10] and [11], a classification of the automorphisms of the graphs of these classes of rings was obtained for all the characteristics.

Matrices being fundamental in the interpretation of graphs, Katja in [8] computed the eigenvalues of graphs associated with zero divisors of finite rings. The study determined computations for nullity, spectrum of $\Gamma(R)$ and the graph product properties. Some independent studies have focused on adjacency matrices of zero divisor graphs of finite completely primary rings. For instance, the research done in [14] was on the characterization of the adjacency and incidence matrices of a class of finite rings of square radical zero. Given an adjacency matrix A and a degree matrix D, the Laplacian matrix is an $n \times n$ matrix L such that $L = D - A$. Some ring classifications have also been done via the Laplacian matrices of their zero divisor graphs. For instance, a study on the ring of integers modulo n was performed in [17] through a research on the nature of their Laplacian eigenvalues. The study involved finding the Laplacian spectrum structures of $\Gamma(\mathbb{Z}_n)$ for $n = p^{N_1}q^{N_2}$ where p is a prime integer greater than q and for positive integers N_1 and N_2. This research was limited only to rings of integers modulo n. Research on distance matrices of classes of finite rings have been done in [12, 15, 18] and [19] among others. In this paper, we focus on a class of 3-radical zero finite completely primary rings and provide an analysis of some graph geometric properties of $\Gamma(R)$. An investigation on the structures of their Adjacency, Laplacian and Distance matrices is also conducted.

2 Preliminaries

The following results will be useful in the sequel.

Theorem 2.1. *[16] Let R be a finite ring with multiplicative identity $1 \neq 0$, whose set of zero divisors forms an additive group J. Then,*

(i) J is the Jacobson radical of R.

(ii) $\mid R \mid = p^{nr}$, and $\mid J \mid = p^{(n-1)r}$ for some prime integer p and some positive integers n, r.

(iii) $J^n = (0)$.

Theorem 2.2. *[6] Let R be a completely primary finite ring of order p^{nr} with unique maximal ideal J such that $\mid R/J \mid = p^r, Char(R) = p^k$. If $R_\circ$ is the maximal Galois subring of R, then there exist $x_1, ..., x_h \in J$ and $\sigma_1, ..., \sigma_h \in Aut(R_\circ)$ such that $R = R_\circ \oplus R_\circ x_i \oplus, ... \oplus R_\circ x_n$ and $x_i r = r^{\sigma_i} x_i$ for every $r \in R_\circ$ and each $i = 1, 2, ..., h$.*

Theorem 2.3. *[4] If $\mathbb{F}$ is the Galois field $GF(p^r)$ and $1 \leq t \leq s^2$ for a fixed $s, t - dimensional$ $\mathbb{F}$-spaces U, V respectively and $(a^k_{ij}) \in \mathbb{M}_{s\times s}(\mathbb{F})$ are t, linearly independent matrices. Then $Aut(R) \cong [\mathbb{M}_{t\times s}(\mathbb{F}) \times (U \oplus V)] \times_{\theta_2} [Aut(\mathbb{F}) \times_{\theta_1} (GL(s, \mathbb{F}) \times GL(t, \mathbb{F}))]$ where $\{\theta_1, ..., \theta_t\}$ is the set of automorphisms of $\mathbb{F}$.*

3 The 3-Radical Zero Completely Primary Finite Rings of characteristic p

The following construction can be obtained in [7].

3.1 Construction I

Given a prime integer p and r, a positive integer, let $R' = \mathbb{F} = GF(p^r)$ be a Galois field of order $q = p^r$. Suppose U, V are finitely generated $\mathbb{F}-$spaces with nonnegative number of elements s and t in the generating sets $\{u_s\}$ and $\{v_t\}$ respectively such that for $t = \frac{s(s+1)}{2}$ and s fixed,
$R = \mathbb{F}_q \oplus \sum_{i=1}^{s} \mathbb{F}_q u_i \oplus \sum_{i,j=1}^{s} \mathbb{F}_q u_i u_j$ is an additive abelian group.
Consider two elements $(a_\circ + \sum_{i=1}^{s} a_i u_i + \sum_{i,j=1}^{s} a_j u_i u_j)$ and $(b_\circ + \sum_{i=1}^{s} b_i u_i + \sum_{i,j=1}^{s} b_j u_i u_j)$ in R and define multiplication on R by:

$$(a_\circ + \sum_{i=1}^{s} a_i u_i + \sum_{i,j=1}^{s} a_j u_i u_j)(b_\circ + \sum_{i=1}^{s} b_i u_i + \sum_{i,j=1}^{s} b_j u_i u_j)$$

$$= a_\circ b_\circ + \sum_{i=1}^{s}((a_\circ b_i^{\sigma_i}) + a_i(b_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(a_\circ b_j + a_j(b_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} \alpha_{ij} a_i (b_j)^{\sigma_i})u_i u_j.$$

where σ_i is an $\mathbb{F}$ automorphism and (α_{ij}) is a t-linearly independent matrix of dimension $s.$
Whenever $\sigma_i = id_{\mathbb{F}},$ an identity automorphism, then, the multiplication defined above turns R into a commutative ring with identity $(1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0})$. Thus, for the rings discussed in this section, we assume that $\sigma_i = id_{\mathbb{F}}.$
The next result which characterizes the structures of the zero divisors $Z(R)$ of R and its proof can be obtained from [5].

Proposition 3.1. *Let R be a ring of construction I and $Z(R)$ be the set of its zero divisors then:*

(i) $Z(R) = \sum_{i=1}^{s} \mathbb{F}_q u_i \oplus \sum_{i,j=1}^{s} \mathbb{F}_q u_i u_j,$

(ii) $(Z(R))^2 = \sum_{i,j=1}^{s} \mathbb{F}_q u_i u_j,$

(iii) $(Z(R))^3 = (0).$

3.2 The Graphs $\Gamma(R)$ and Matrices obtained from Classes of Rings in Construction I

Here we determine some graph algebraic properties, formulate the matrices from the graphs and investigate the matrix algebraic properties.

Proposition 3.2. *Let R be a ring of construction I and the invariants $p,\ s$ and r be positive integers. Then, the following properties hold:*

(i) $\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+3s)}{2})r} - 1.$

(ii) $\Gamma(R)$ *is an incomplete graph.*

(iii) The $diam((\Gamma(R)) = 2.$

(iv) The minimum degree, $\delta(\Gamma(R)) = p^{(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s)}{2}-1)r} - 1.$

(v) The $girth(\Gamma(R)) = 3.$

Proof. (i) Since $char(R) = p, pu_i = pu_iu_j = 0,$ and we have that $\mid \mathbb{F}_q u_i \mid= p^r, \mid \mathbb{F}_q u_i u_j \mid= p^r$, for any $i, j, = 1, 2, \cdots, s$ we obtain $\mid Z(R) \mid= p^{(\frac{(s^2+3s)}{2})r}$. Therefore, $\mid Z(R)^* \mid=\mid Z(R)\backslash\{0\} \mid= p^{(\frac{(s^2+3s)}{2})r} - 1$. Since $\mid Z(R)^* \mid=\mid V(\Gamma(R)) \mid$, it follows that $\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+3s)}{2})r} - 1.$

(ii) Easily follows from the fact that $(Z(R))^2 \neq (0)$.

(iii) From (ii), $\Gamma(R)$ is incomplete and with the fact that $Ann(Z(R)) = (Z(R))^2$, the result follows.

(iv) Let $V = \{v_1, v_2, \cdots, v_{p^{(\frac{(s^2+3s)}{2})r}-1}\}$ be the whole vertex set of $\Gamma(R)$. Let $K, S \subseteq V$ such that $K \subseteq ann(Z(R))^*$. This implies $\mid K \mid= p^{(\frac{(s^2+3s)}{2}-1)r} - 1 \Longrightarrow \mid S \mid= (p^{(\frac{(s^2+3s)}{2})r} - 1) - (p^{(\frac{(s^2+3s)}{2}-1)r} - 1) = p^{(\frac{(s^2+3s)}{2})r} - 1 - p^{(\frac{(s^2+3s)}{2}-1)r} + 1 \Rightarrow p^{(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s)}{2}-1)r}$.
Therefore, $\delta(\Gamma(R)) = p^{(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s)}{2}-1)r} - 1$ due to the minimal degree of the elements of S and for the avoidance of self loop for each vertex $v \in S$.

(v) Taking two vertices $v_1, v_2 \in Z(R) - (Z(R))^2$ where $v_1v_2 = 0$, clearly each v_1, v_2 is adjacent to some $v_3 \in (Z(R))^2$. Thus $v_1 - v_2 - v_3 - v_1$ is the least polygon in $\Gamma(R)$.

□

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Next, we investigate the properties of the Adjacency, Laplacian and Distance matrices.

Proposition 3.3. *Let R be a ring given in construction I and $\Gamma(R)$ be its zero divisor graph. Then, the adjacency matrix associated with $\Gamma(R)$ is of trace 0 with a spectral radius p^r+1. Furthermore, for an adjacency matrix $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$, the following properties hold:*

(i) $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ *is symmetric.*

(ii) $rank\ ([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = p^{(\frac{(s^2+3s)}{2})r} - 2p^r$.

(iii) $Tr\ ([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0$.

(iv) $Det([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0$.

(v) The eigenvalues
$$\lambda\ ([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \begin{cases} \pm\sqrt{2} \text{ and } 0. & \text{when } p=2; \\ 0, \quad \text{of multiplicity } 2p^r-1; & \\ -1, & \\ -p^r, \quad \text{and} & \text{when } p \neq 2. \\ p^r+1. & \end{cases}$$

Proof. (i) Since every row vector
$(a_{11}, a_{12}, \cdots, a_{1(p^{(\frac{(s^2+3s)}{2})r}-1)}), (a_{21}, a_{22}, \cdots, a_{2(p^{(\frac{(s^2+3s)}{2})r}-1)}), \cdots,$
$(a_{(p^{(\frac{(s^2+3s)}{2})r}-1)1}, a_{(p^{(\frac{(s^2+3s)}{2})r}-1)2}, \cdots, a_{(p^{(\frac{(s^2+3s)}{2})r}-1)(p^{(\frac{(s^2+3s)}{2})r}-1)})$ of $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ is a reflection of the corresponding element through the leading diagonal to every column

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{(p^{(\frac{(s^2+3s)}{2})r}-1)1} \end{pmatrix} \cdots \begin{pmatrix} a_{1(p^{(\frac{(s^2+3s)}{2})r}-1)} \\ a_{2(p^{(\frac{(s^2+3s)}{2})r}-1)} \\ \vdots \\ a_{(p^{(\frac{(s^2+3s)}{2})r}-1)(p^{(\frac{(s^2+3s)}{2})r}-1)} \end{pmatrix},$$

which implies that $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1} = [A]^T_{p^{(\frac{(s^2+3s)}{2})r}-1}$. Hence the symmetry of $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$.

(ii) Upon carrying out a row operation on

$$[A]_{p^{(\frac{(s^2+3s)}{2})^r-1}} = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & & 1 \\ 1 & 0 & 1 & 1 & \cdots & & 1 \\ \vdots & 0 & 1 & 1 & \cdots & & 1 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0_{p^{(\frac{(s^2+3s)}{2})^r-2p^r}} \\ \vdots & \vdots & \vdots & 0 & \cdots & & \vdots \\ 0_{p^{(\frac{(s^2+3s)}{2})^r-1}} & 0 & \cdots & \cdots & \cdots & & 0_{p^{(\frac{(s^2+3s)}{2})^r-1}} \end{pmatrix},$$

we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & 0 & \cdots & & 0 \\ \vdots & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & 0 & 0 & 1 & 0 & \cdots & 0_{p^{(\frac{(s^2+3s)}{2})^r-2p^r}} \\ \vdots & \vdots & \vdots & 0 & \cdots & & \vdots \\ 0_{p^{(\frac{(s^2+3s)}{2})^r-1}} & 0 & \cdots & \cdots & \cdots & & 0_{p^{(\frac{(s^2+3s)}{2})^r-1}} \end{pmatrix}.$$

Let $V = \{v_1, v_2, v_3, \cdots, v_{p^{(\frac{(s^2+3s)}{2})^r-2}}\}$ be the linearly independent set of vectors such that

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, v_{p^{(\frac{(s^2+3s)}{2}-1)^r}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1_{p^{(\frac{(s^2+3s)}{2})^r-2p^r}} \\ 0 \\ \vdots \\ 0_{p^{(\frac{(s^2+3s)}{2})^r-1}} \end{pmatrix}.$$

Clearly, the set spans the matrix space implying that the $rank([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}) = p^{(\frac{(s^2+3s)}{2})^r} - 2p^r$.

(iii) Given that $[A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}$ is the adjacency matrix for $\Gamma(R)$, it is justifiable that $Tr([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}) = a_{11} + a_{22} + a_{33} + \cdots + a_{(p^{(\frac{(s^2+3s)}{2})^r-1})(p^{(\frac{(s^2+3s)}{2})^r-1})}$.

Since $\Gamma(R)$ is a simple graph with no self loop for $v_i = 1, 2, \cdots, p^{(\frac{(s^2+3s)}{2})^r} - 1$, the leading diagonal entries are 0. Therefore, $\sum_{i=1}^{p^{(\frac{(s^2+3s)}{2})^r-1}} a_{ii} = 0$.

Further, since $Tr([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}) = \sum_{j=1}^{p^{(\frac{(s^2+3s)}{2})^r-1}} \lambda_j$. Therefore, the sum of eigenvalues $\lambda_j = -1 - p^r + p^r + 1 = 0$ and the spectral radius is $p^r + 1$.

(iv) Consider $[B]_{p^{(\frac{(s^2+3s)}{2})^r-1}} = [A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}[C]^T_{p^{(\frac{(s^2+3s)}{2})^r-1}}$ where $[C]_{p^{(\frac{(s^2+3s)}{2})^r-1}}$ is the cofactor matrix of $[A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}$ then $b_{ij} = \sum_k a_{ik}c_{jk}$ for c_{jk} is the jk minor of $[A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}$. If $i = j$, it corresponds to the determinant computation of $[A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}$ along the i^{th} row. Hence $b_{ii} = det([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}})$.
If $i \neq j$, this corresponds to the determinant computation of a matrix equal to $[A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}$ except that the row j has been overwritten by the contents of i^{th} row. But the determinant of a matrix with duplicated row is 0, hence $b_{ij} = 0$. $\Rightarrow [A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}[C]^T_{p^{(\frac{(s^2+3s)}{2})^r-1}} = det([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}})$
If $det([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}) \neq 0$ then we can write

$$A\frac{C^T}{det([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}})} = I \Leftrightarrow [A]^{-1}_{p^{(\frac{(s^2+3s)}{2})^r-1}} = \frac{C^T}{det([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}})} = \frac{Adj[A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}}{det([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}})}.$$

i.e $det([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}) \neq 0 \Rightarrow [A]^{-1}_{p^{(\frac{(s^2+3s)}{2})^r-1}}$ exists . Consequently, if $[A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}$ is singular then $det([A]_{p^{(\frac{(s^2+3s)}{2})^r-1}}) = 0$.

(v) For $p = 2$, we can obtain the eigenvalues by solving the equation $\mid \lambda I - A \mid = 0 \Longrightarrow \lambda^3 - 2\lambda^2 = 0 \Rightarrow \lambda(\lambda^2 - 2) = 0 \Rightarrow \lambda = 0$ and $\lambda = \pm\sqrt{2}$.
For $p \neq 2$,
$\mid \lambda I - A \mid = 0$ results to a characteristic equation of the form $\lambda^{2p^r-1}(-(p^r+1)+\lambda)(1+\lambda)(p^r+\lambda) = 0$. Solving for λ in every factor results to $\lambda^{2p^r-1} = 0 \Longrightarrow \lambda = 0$ of multiplicity $2p^r - 1$. For the second factor, $(-(p^r+1)+\lambda) = 0 \Longrightarrow \lambda = p^r + 1$. For $(1+\lambda) = 0, \lambda = -1$ and $(p^r + \lambda) = 0 \Rightarrow \lambda = -p^r$.

□

Proposition 3.4. *Consider* $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$*, the adjacency matrix associated with* $\Gamma(R)$ *where* R *is a ring of construction I. Then for any fixed* $s \neq 1, r \in \mathbb{Z}^+$ *and* $p,$ *prime integer, the following properties hold:*

(i) $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ *is symmetric.*

(ii) $rank\ ([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = p^{(\frac{(s^2+3s)}{2}-1)r}.$

(iii) $Tr\ ([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0.$

(iv) $Det([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0.$

(v) *For p=2,* $\lambda[A]_{p^{(\frac{(s^2+3s)}{2})r}-1} = \begin{cases} -1, & \text{of multiplicity } p^r; \\ 0, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2}-1)r} - 1; \\ 1 \pm \sqrt{p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r} + 1}, & . \end{cases}$

For $p \geq 3,$
The eigenvalues $\lambda[A]_{p^{(\frac{(s^2+3s)}{2})r}-1} =$

$$\begin{cases} -1, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2}-1)r} - 2; \\ 0, & \text{of multiplicity } 2p^{(\frac{(s^2+3s)}{2}-1)r} - 1; \\ \frac{p^{(\frac{(s^2+3s)}{2}-1)r}-2\pm\sqrt{9p^{(\frac{(s^2+3s)}{2}+1)r}-4p^{(\frac{(s^2+3s)}{2})r}-8p^{(\frac{(s^2+3s)}{2}-1)r}+4}}{2}, & . \end{cases}$$

Proof. The proof for (i) to (iv) have similar steps to the ones in Proposition 3.3.
(v) For $p = 2$, the equation $\mid [A]_{p^{(\frac{(s^2+3s)}{2})r}-1} - \lambda I_{p^{(\frac{(s^2+3s)}{2})r}-1} \mid = 0$ yields the characteristic equation for the adjacency matrix $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$. Let the eigenvalues of $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ be $\lambda_1, \lambda_2, \cdots, \lambda_{p^{(\frac{(s^2+3s)}{2})r}-1}$. We can obtain the characteristic equation of the adjacency matrix as

$$-\lambda^{p^{(\frac{(s^2+3s)}{2}-1)r}-1}(1+\lambda)^{p^r}(\lambda^2 - 2\lambda - (p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r})) = 0.$$

$\Rightarrow -\lambda^{p^{(\frac{(s^2+3s)}{2}-1)r}-1} = 0 \Rightarrow \lambda = 0$ of multiplicity $p^{(\frac{(s^2+3s)}{2}-1)r} - 1, (1+\lambda)^{p^r} = 0 \Rightarrow \lambda = -1$ of multiplicity p^r. By solving the quadratic equation
$\lambda^2 - 2\lambda - (p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r}) = 0$ in the last factor, we obtain

$$\lambda = \frac{2 \pm \sqrt{4 + 4p^{(\frac{(s^2+3s)}{2})r} + 4p^{(\frac{(s^2+3s)}{2}-1)r}}}{2} = \frac{2 \pm \sqrt{4(p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r} + 1)}}{2}$$

$$= \frac{2 \pm 2\sqrt{p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r} + 1}}{2} = 1 \pm \sqrt{p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r} + 1}.$$

For $p \neq 2$, we generally obtain the polynomial equation which when factorized results to $\lambda^{2p^{(\frac{(s^2+3s)}{2}-1)r}-1}(1+\lambda)^{p^{(\frac{(s^2+3s)}{2}-1)r}-2}((\lambda^2 - (p^{(\frac{(s^2+3s)}{2}-1)r} - 2)\lambda - (2p^{(\frac{(s^2+3s)}{2}+1)r} - p^{(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s)}{2}-1)r})) = 0$. Solving the equation gives the eigenvalues as $\lambda^{2p^{(\frac{(s^2+3s)}{2}-1)r}-1} = 0 \Rightarrow \lambda = 0$ of multiplicity $2p^{(\frac{(s^2+3s)}{2}-1)r} - 1$. The second factor $(1+\lambda)^{p^{(\frac{(s^2+3s)}{2}-1)r}-2} = 0 \Rightarrow \lambda = -1$ of multiplicity $p^{(\frac{(s^2+3s)}{2}-1)r} - 2$. Finally, solving the quadratic part results to

$$\lambda = \frac{(p^{(\frac{(s^2+3s)}{2}-1)r} - 2) \pm \sqrt{(p^{(\frac{(s^2+3s)}{2}-1)r} - 2)^2 + 4(2p^{(\frac{(s^2+3s)}{2}+1)r} - p^{(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s)}{2}-1)r})}}{2}.$$

Expanding the discriminant yields the expression
$p^{(\frac{(s^2+3s)}{2}+1)r} - 4p^{(\frac{(s^2+3s)}{2}-1)r} + 4 + 8p^{(\frac{(s^2+3s)}{2}+1)r} - 4p^{(\frac{(s^2+3s)}{2})r} - 4p^{(\frac{(s^2+3s)}{2}-1)r}$ so that

$$\lambda = \frac{(p^{(\frac{(s^2+3s)}{2}-1)r} - 2) \pm \sqrt{9p^{(\frac{(s^2+3s)}{2}+1)r} - 4p^{(\frac{(s^2+3s)}{2})r} - 8p^{(\frac{(s^2+3s)}{2}-1)r} + 4}}{2}.$$

□

Proposition 3.5. *Consider* $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$*, the Laplacian matrix associated with* $\Gamma(R)$ *of the ring in Construction I. Then for any positive integer* r*, prime integer* p*, the following properties hold:*

(i) $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ *is symmetric.*

(ii) $rank([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = p^{(\frac{(s^2+3s)}{2})r} - 2.$

(iii) $Tr([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \begin{cases} 4, & \textit{when } s=1, p=2; \\ (p^r-1)(2p^{(\frac{(s^2+3s)}{2})r} - p^r - 2), & \textit{for any } s,p \geq 2. \end{cases}$

(iv) $Det([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0.$

(v) The eigenvalues $\lambda[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ *are* $0, 1$ *and* 3 *when* $s=1, p=2$.

For any $s,p \geq 2$ *the eigenvalues* $\lambda[L]_{p^{(\frac{(s^2+3s)}{2})r}-1} = \begin{cases} 0, & \\ p^{(\frac{(s^2+3s)}{2})r} - 1, & \textit{of multiplicity } p^r - 1; \\ p^r - 1, & \textit{of multiplicity } 2p^r - 1. \end{cases}$

Proof. (i) Can be drawn from the previous proposition since the steps are similar.
(ii) We conduct a row operation on $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ to obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & . & . & -1 \\ 0 & 1 & 0 & 0 & . & . & -1 \\ 0 & . & \ddots & 0 & . & . & -1 \\ \vdots & . & . & 1_{p^{(\frac{(s^2+3s)}{2})r}-2} & 0 & 1 & -1 \\ 0 & 0 & . & . & . & 0 & 0 \end{pmatrix}.$$

This results to $p^{(\frac{(s^2+3s)}{2})r} - 2$ non-zero rows in $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$, hence its rank.
(iii) For $p=2$ and $s=1$, the $\Gamma(R)$ obtained has only 1 vertex of maximum degree 2 and 2 vertices of minimum degree 1. This leads to a Laplacian matrix of order 3×3 whose main diagonal entries are $2, 1$ and 1 hence a trace of 4.
For any $s,p \geq 2$,
$\mid Z(R)^* \mid = p^{(\frac{(s^2+3s)}{2})r} - 1$ and each $v_i \in Ann(Z(R)^*)$ has degree $p^{(\frac{(s^2+3s)}{2})r} - 2$ and $\mid Ann(Z(R)^*) \mid = p^r - 1$. Therefore any $v_j \notin Ann(Z(R)^*)$ is of degree $p^r - 1$ because every such $v_j \notin Ann(Z(R)^*)$ is only adjacent to $v_i \in Ann(Z(R)^*)$.
Partitioning the vertices of $\Gamma(R)$ into disjoint subsets V_1 and V_2 such that
$V_1 = \{v_j | v_j \notin Ann(Z(R)^*)\}$ and $V_2 = \{v_i | v_i \in Ann(Z(R)^*)\}$,
$\mid V_2 \mid = p^r - 1$ and $\mid V_1 \mid = p^{(\frac{(s^2+3s)}{2})r} - 1 - (p^r - 1) = p^{(\frac{(s^2+3s)}{2})r} - p^r$. Since the trace, $Tr([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \sum_{i=1}^{p^{(\frac{(s^2+3s)}{2})r}-1} l_{ii}$, where every l_{ii} is an element of the diagonal matrix $[D]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ whose entries are degrees of vertices in $\Gamma(R)$, we have that $Tr([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = (p^{(\frac{(s^2+3s)}{2})r} - 2)(p^r - 1) + (p^r -$

$1)(p^{(\frac{(s^2+3s)}{2})r} - p^r)$. This results to $(p^r-1)(p^{(\frac{(s^2+3s)}{2})r} - 2 + p^{(\frac{(s^2+3s)}{2})r} + p^r) = (p^r-1)(2p^{(\frac{(s^2+3s)}{2})r} - p^r - 2)$.
(iv) Steps in obtaining the singularity of $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ are similar to the one in proposition 3.4.
(v) For $p = 2, s = 1$, the eigenvalues for the 3×3 Laplacian matrix are easy to obtain.
When $p \geq 2$ for any s, the equation $|\ (\lambda I_{p^{(\frac{(s^2+3s)}{2})r}-1} - [L]_{p^{(\frac{(s^2+3s)}{2})r}-1})\ | = 0$ gives the characteristic polynomial equation of the form $-\lambda((-(p^{(\frac{(s^2+3s)}{2})r} - 1) + \lambda)^{p^r-1}(-(p^r-1)+\lambda)^{2p^r-1}) = 0$. On solving each factor, we obtain $-\lambda = 0 \implies \lambda = 0$. For the factor $(-(p^{(\frac{(s^2+3s)}{2})r} - 1) + \lambda)^{p^r-1} = 0$, we have that $\lambda = (p^{(\frac{(s^2+3s)}{2})r} - 1)$ of multiplicity $p^r - 1$. Finally, $(-(p^r-1)+\lambda)^{2p^r-1} = 0 \implies \lambda = p^r - 1$, of multiplicity $2p^r - 1$. This establishes (v). □

Next, we take an analysis of the distance matrix of $\Gamma(R)$ of the ring in Construction I. Recall that the distance matrix of a graph G having n vertices is a symmetric matrix $[d_{ij}]$ whose entry d_{ij} is defined as $d_{ij} = \begin{cases} d(v_i, v_j), & \text{if } i \neq j; \\ 0, & \text{if } i = j. \end{cases}$

The following result describes the matrix algebraic properties of $[d_{ij}]$ of this class of rings.

Proposition 3.6. *Consider $[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}$, distance matrix associated with $\Gamma(R)$ of R in Construction I. For $r \in \mathbb{Z}^+, p$ prime,*

(i) $Det([d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \begin{cases} 4, & \textit{when } p = 2, s = 1; \\ -(p^{(\frac{(s^2+3s)}{2})r} - 1)^2, & \textit{for any } s, p \geq 2. \end{cases}$

(ii) $Tr([d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0.$

(iii) $rank([d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \begin{cases} 3, & \textit{when } p = 2, s = 1; \\ p^{(\frac{(s^2+3s)}{2})r} - 1, & \textit{for any } s, p \geq 2. \end{cases}$.

(iv) When $p = 2, s = 1$ the eigenvalues $\lambda[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ are $1 \pm \sqrt{3}$ and 2

For any $s, p \geq 2$, the eigenvalues $\lambda[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}$

$$= \begin{cases} -1, & \\ 1 - p^r, & \textit{multiplicity } p^{(\frac{(s^2+3s)}{2})r} + (p^r + 1); \\ \frac{1}{2}((p^{(\frac{(s^2+3s)}{2})r} + 2) \pm \sqrt{p^{(\frac{2(s^2+3s)}{2})r} + 4p^{(\frac{(s^2+3s)}{2})r} + 4p^r}) & . \end{cases}$$

Proof. (i) When $p = 2, s = 1$, we obtain the distance matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.

Expanding the matrix along the first row, we clearly obtain the determinant to be 4.
When $p \geq 2$ for any fixed s, we obtain the distance matrix to be of the form

$$\begin{pmatrix} 0 & 1 & \cdots & \cdots & \cdots & 1 & 1_{p^{(\frac{(s^2+3s)}{2})^r}-1} \\ 1 & 0 & 1 & \cdots & 1 & \cdots & \cdots \\ \vdots & \cdots & \ddots & & & & 1_{p^{(\frac{(s^2+3s)}{2})^r}-1} \\ 1 & 1 & 2 & 0 & 2 & \cdots & 2 \\ \vdots & \cdots & \vdots & \cdots & \ddots & & \vdots \\ 1 & 1 & \cdots & 2 & \cdots & & 0 \end{pmatrix}.$$

Expanding along the first row, we obtain the determinant to be $-(p^{(\frac{(s^2+3s)}{2})^r} - 1)(p^{(\frac{(s^2+3s)}{2})^r} - 1) = -(p^{(\frac{(s^2+3s)}{2})^r} - 1)^2$.
(ii) Due to the fact that $d(v_i, v_i) = 0$, this results to a distance matrix $[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})^r}-1}$ with $0'$s entirely in the main diagonal. Therefore, $Tr([d_{ij}]_{p^{(\frac{(s^2+3s)}{2})^r}-1}) = \sum_{i=1}^{p^{(\frac{(s^2+3s)}{2})^r}-1} d(v_i, v_i) = 0$.
(iii) When $p = 2, s = 1$, the rank of the matrix in (i) is clearly 3.
When $p \geq 2$ for any fixed s, we can obtain the rank of the distance matrix $[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})^r}-1}$ by conducting a row operation on it which reduces to the echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 2 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 2 \\ \vdots & & \ddots & & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & \cdots & 0 & 2 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & -1 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & & 0 & 1 & -1_{p^{(\frac{(s^2+3s)}{2})^r}-2} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \cdots & 1_{p^{(\frac{(s^2+3s)}{2})^r}-1} \end{pmatrix}.$$

From the reduced echelon form above, there are $p^{(\frac{(s^2+3s)}{2})^r} - 1$ linearly independent vectors which span the row space of $[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})^r}-1}$ resulting to $p^{(\frac{(s^2+3s)}{2})^r} - 1$ nonzero rows, hence its rank.

(iv) Solving the equation $\mid \lambda I - [d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1} \mid = 0$, results to the characteristic equation $(1+\lambda)((p^r-1)+\lambda)^{p^{(\frac{(s^2+3s)}{2})r}+(p^r+1)}(\lambda^2-(p^{(\frac{(s^2+3s)}{2})r}+2)\lambda-(p^r-1)) = 0$. We obtain the eigenvalues by finding the solution for λ in every factor of the equation as follows: Clearly, $(\lambda+1) = 0 \Longrightarrow \lambda = -1$. Further, $((p^r-1)+\lambda)^{p^{(\frac{(s^2+3s)}{2})r}+(p^r+1)} = 0$, we obtain $\lambda = 1-p^r$ of multiplicity $p^{(\frac{(s^2+3s)}{2})r} + (p^r+1)$. Finally, for the quadratic part $(\lambda^2 - (p^{(\frac{(s^2+3s)}{2})r}+2)\lambda - (p^r-1)) = 0$, we can obtain
$\lambda = \frac{1}{2}((p^{(\frac{(s^2+3s)}{2})r}+2) \pm \sqrt{(p^{(\frac{(s^2+3s)}{2})r}+2)^2 + 4(p^r-1)})$
which on expansion yields $\lambda = \frac{1}{2}((p^{(\frac{(s^2+3s)}{2})r}+2) \pm \sqrt{p^{(\frac{2(s^2+3s)}{2})r} + 4p^{(\frac{(s^2+3s)}{2})r} + 4 + 4(p^r-1)})$ and simplifies to $\lambda = \frac{1}{2}((p^{(\frac{(s^2+3s)}{2})r}+2) \pm \sqrt{p^{(\frac{2(s^2+3s)}{2})r} + 4p^{(\frac{(s^2+3s)}{2})r} + 4p^r})$. □

4 The 3-Radical Zero Finite Completely Primary Rings of characteristic p^2

4.1 Construction II

Let $R' = GR(p^{2r}, p^2)$ be a Galois ring of order p^{2r} and of characteristic p^2. Let U and V be finitely generated R'-modules with $\{u_1, u_2, \cdots, u_s\}$ and $\{v_1, v_2, \cdots, v_t\}$ being the generating set such that the nonnegative integers s, t are the number of elements in the generating sets. Then for fixed s and $t = \frac{s(s+1)}{2}$, $R = R' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$ is an additive abelian group. Define multiplication in R by

$$(a_\circ + \sum_{i=1}^{s} a_iu_i + \sum_{i,j=1}^{s} a_ju_iu_j)(b_\circ + \sum_{i=1}^{s} b_iu_i + \sum_{i,j=1}^{s} b_ju_iu_j) =$$

$$a_\circ b_\circ + \sum_{i=1}^{s}((a_\circ + pR')b_i + a_i(b_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(a_\circ b_j + a_j(b_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} \alpha_{ij}^k a_i(b_j)^{\sigma_i})u_iu_j.$$

R is thus turned by the multiplication into a commutative ring with identity $(1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0})$.

For the rings discussed in this section, we shall consider $\sigma_i = id_{R'}$. The following Proposition and its proof can be obtained from [5].

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Case I: when $pu_i = 0$

Proposition 4.1. *Consider R from Construction II and let $pu_i = 0$. Then zero divisors set $Z(R)$ satisfies the following properties:*

(i) $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$.

(ii) $(Z(R))^2 = \sum_{i,j=1}^{s} R'u_iu_j$.

(iii) $(Z(R))^3 = (0)$.

4.2 The Graphs $\Gamma(R)$ and Matrices obtained from Classes of Rings in Construction II

Proposition 4.2. *Consider R from Construction II. Then for p prime, $r \in \mathbb{Z}^+$ and $pu_i = 0$, $\Gamma(R)$ has the following properties:*

(i) $\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+3s+2)}{2})r} - 1$.

(ii) $\Gamma(R)$ *is an incomplete graph.*

(iii) $diam((\Gamma(R)) = 2$.

(iv) The minimum degree, $\delta(\Gamma(R)) = p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r} - 1$.

(v) $girth(\Gamma(R)) = 3$.

Proof. (i) Since $char(R) = p^2,\ pu_i = 0$, and we have that $\mid R'u_i \mid= p^r, \mid R'u_iu_j \mid= p^r, \mid pR' \mid= p^r$ which is true for every $i, j = 1, \cdots s$. We obtain $\mid Z(R) \mid= p^{(\frac{(s^2+3s+2)}{2})r}$. Therefore, $\mid Z(R)\backslash\{0\} \mid= p^{(\frac{(s^2+3s+2)}{2})r} - 1$ since $\mid Z(R)^* \mid=\mid V(\Gamma(R)) \mid\Longrightarrow\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+3s+2)}{2})r} - 1$.
(ii) Follows eaasily from the fact that $(Z(R))^2 \neq (0)$.
(iii) From the incompleteness of $\Gamma(R)$ in (ii), $Ann(Z(R)) = (Z(R))^2$, there exist some two non adjacent vertices $x, y \in V(\Gamma(R))$ so that for some $z \in Ann(Z(R))$, the supremum distance $d\{x, y\} = 2$ hence the diameter.
(iv) Let $V = \{v_1, v_2, \cdots, v_{p^{(\frac{(s^2+3s+2)}{2})r}-1}\}$ be the vertex set for $\Gamma(R)$. Let $K, S \subseteq V$ such that $K \subseteq ann(Z(R))^*$. This implies $\mid K \mid= p^{(\frac{(s^2+3s)}{2})r} - 1$ and $\mid S \mid= (p^{(\frac{(s^2+3s+2)}{2})r} - 1) - (p^{(\frac{(s^2+3s)}{2})r} - 1) =$

$p^{(\frac{(s^2+3s+2)}{2})r} - 1 - p^{(\frac{(s^2+3s)}{2})r} + 1$ which simplifies to $p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}$. Therefore, $\delta(\Gamma(R)) = (p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}) - 1 = p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r} - 1$ due to the minimal degree of the elements of S and for the avoidance of self loop for each vertex $v \in S$. Hence minimum degree $\delta(\Gamma(R))$.
(v) Follows from Proof (v) in Proposition 3.2. □

Proposition 4.3. *Consider* $[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$*, the adjacency matrix associated with* $\Gamma(R)$ *for* R *in Construction II. Then for any fixed* $s, r \in \mathbb{Z}^+$ *and* p *prime,*

(i) $[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ *is symmetric.*

(ii) $rank\ ([A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = p^{(\frac{(s^2+3s)}{2})r} - 1.$

(iii) $Tr\ ([A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = 0.$

(iv) $Det([A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = 0.$

(v) For $p = 2,$ *the eigenvalues* $\lambda[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$

$$= \begin{cases} -1, & \text{of multiplicity } p^r; \\ 0, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2})r} - 1; \\ 1 \pm \sqrt{p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} + 1}, & . \end{cases}$$

For $p \neq 2,$ *eigenvalues* $\lambda[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$

$$= \begin{cases} -1, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2})r} - 2; \\ 0, & \text{of multiplicity } 2p^{(\frac{(s^2+3s)}{2})r} - 1; \\ \frac{(p^{(\frac{(s^2+3s)}{2})r}-2) \pm \sqrt{p^{2(\frac{(s^2+3s)}{2})r} - 8p^{(\frac{(s^2+3s)}{2})r} + 4p^{(\frac{(s^2+3s+2)}{2})r} + 4}}{2}, & . \end{cases}$$

Proof. (i) The steps in the proof for (i),(iii) and (iv) are clear. We provide proofs for (ii) and (v) as follows:

(ii) Upon carrying out a row operation on

$$[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1} = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & & 1 \\ 1 & 0 & 1 & 1 & \cdots & & 1 \\ \vdots & \vdots & 0 & 1 & \cdots & & \vdots \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0_{p^{(\frac{(s^2+3s)}{2})r}} \\ \vdots & \vdots & \vdots & 0 & \cdots & & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & & 0 \end{pmatrix},$$

we obtain its reduced echelon form matrix as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & 0 & \cdots & & 0 \\ \vdots & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & 0 & 0 & 1 & 0 & \cdots & 0_{p^{(\frac{(s^2+3s)}{2})r}} \\ \vdots & \vdots & \vdots & 0 & \cdots & & \vdots \\ 0_{p^{(\frac{(s^2+3s+2)}{2})r}-2} & 0 & \cdots & \cdots & \cdots & & 0_{p^{(\frac{(s^2+3s+2)}{2})r}-1} \end{pmatrix}.$$

Let $V = \{v_1, v_2, v_3, \cdots, v_{p^{(\frac{(s^2+3s)}{2})r}}\}$ be the linearly independent set of vectors such that $v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, v_{p^{(\frac{(s^2+3s)}{2})r}} = \begin{pmatrix} 0 \\ \vdots \\ 1_{p^{(\frac{(s^2+3s)}{2})r}-1} \\ 0 \\ \vdots \\ 0_{p^{(\frac{(s^2+3s+2)}{2})r}-1} \end{pmatrix}$. Clearly, the set V spans the whole of the matrix space. Therefore the $rank([A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = p^{(\frac{(s^2+3s)}{2})r} - 1.$

(v) For $p = 2$, $|\ \lambda I_{p^{(\frac{(s^2+3s+2)}{2})r}-1} - [A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}\ | = 0$ yields the characteristic equation for the adjacency matrix $[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$. Let the eigenvalues of $[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ be $\lambda_1, \lambda_2, \cdots, \lambda_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$. We can obtain the characteristic equation of the adjacency matrix as $-\lambda^{p^{(\frac{(s^2+3s)}{2})r}-1}(1+\lambda)^{p^r}(\lambda^2 - 2\lambda - (p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r})) = 0. \Rightarrow -\lambda^{p^{(\frac{(s^2+3s)}{2})r}-1} = 0 \Rightarrow \lambda = 0$ of multiplicity $p^{(\frac{(s^2+3s)}{2})r}-1,\ (1+\lambda)^{p^r} = 0 \Rightarrow \lambda = -1$ of multiplicity p^r. By solving the quadratic equation

$\lambda^2-2\lambda-(p^{(\frac{(s^2+3s+2)}{2})^r}+p^{(\frac{(s^2+3s)}{2})^r})=0$ in the last factor, we obtain $\lambda=\frac{2\pm\sqrt{4+4p^{(\frac{(s^2+3s+2)}{2})^r}+4p^{(\frac{(s^2+3s)}{2})^r}}}{2}=\frac{2\pm\sqrt{4(p^{(\frac{(s^2+3s+2)}{2})^r}+p^{(\frac{(s^2+3s)}{2})^r}+1)}}{2}=\frac{2\pm2\sqrt{p^{(\frac{(s^2+3s+2)}{2})^r}+p^{(\frac{(s^2+3s)}{2})^r}+1}}{2}=1\pm\sqrt{p^{(\frac{(s^2+3s+2)}{2})^r}+p^{(\frac{(s^2+3s)}{2})^r}+1}.$

For $p\neq 2,$
we generally obtain the polynomial equation which when factorized results to $\lambda^{2p^{(\frac{(s^2+3s)}{2})^r-1}}(1+\lambda)^{p^{(\frac{(s^2+3s)}{2})^r-2}}((\lambda^2-(p^{(\frac{(s^2+3s)}{2})^r}-2)\lambda-(p^{(\frac{(s^2+3s+2)}{2})^r}-p^{(\frac{(s^2+3s)}{2})^r}))=0$. Solving the equation gives the eigenvalues as $\lambda^{2p^{(\frac{(s^2+3s)}{2})^r-1}}=0\Rightarrow\lambda=0$ of multiplicity $2p^{(\frac{(s^2+3s)}{2})^r}-1$. The second factor $(1+\lambda)^{p^{(\frac{(s^2+3s)}{2})^r-2}}=0\Rightarrow\lambda=-1$ of multiplicity $p^{(\frac{(s^2+3s)}{2})^r}-2$. Finally, solving the quadratic part results to

$\lambda=\frac{(p^{(\frac{(s^2+3s)}{2})^r}-2)\pm\sqrt{(p^{(\frac{(s^2+3s)}{2})^r}-2)^2+4(p^{(\frac{(s^2+3s+2)}{2})^r}-p^{(\frac{(s^2+3s)}{2})^r})}}{2}.$

Expanding the discriminant yields the expression $p^{2(\frac{(s^2+3s)}{2})^r}-4p^{(\frac{(s^2+3s)}{2})^r}+4+4p^{(\frac{(s^2+3s+2)}{2})^r}-4p^{(\frac{(s^2+3s)}{2})^r}$ so that

$\lambda=\frac{(p^{(\frac{(s^2+3s)}{2})^r}-2)\pm\sqrt{p^{2(\frac{(s^2+3s)}{2})^r}-8p^{(\frac{(s^2+3s)}{2})^r}+4p^{(\frac{(s^2+3s+2)}{2})^r}+4}}{2}.$

□

Proposition 4.4. *Consider* $[L]_{p^{(\frac{(s^2+3s+2)}{2})^r}-1}$, *the Laplacian matrix associated with* $\Gamma(R)$ *in Construction II such that* $pu_i=0$. *Then for* $r\in\mathbb{Z}^+, p$ *prime and for a fixed* $s,$

(i) $[L]_{p^{(\frac{(s^2+3s+2)}{2})^r}-1}$ *is symmetric.*

(ii) $rank\ ([L]_{p^{(\frac{(s^2+3s+2)}{2})^r}-1})=p^{(\frac{(s^2+3s+2)}{2})^r}-2.$

(iii) $Tr([L]_{p^{(\frac{(s^2+3s+2)}{2})^r}-1})=2p^{(\frac{2(s^2+3s)}{2})^r}-2p^{2(\frac{(s^2+3s)}{2})^r}-2p^{(\frac{(s^2+3s+2)}{2})^r}-p^{(\frac{(s^2+3s)}{2})^r}+2.$

(iv) $Det([L]_{p^{(\frac{(s^2+3s+2)}{2})^r}-1})=0.$

(v) The eigenvalues $\lambda[L]_{p^{(\frac{(s^2+3s+2)}{2})^r}-1}=\begin{cases}0,\\ p^{(\frac{(s^2+3s+2)}{2})^r}-1, & \textit{of multiplicity } p^{(\frac{(s^2+3s)}{2})^r}-1;\\ p^{(\frac{(s^2+3s)}{2})^r}-1, & \textit{of multiplicity } p^{(\frac{(s^2+3s)}{2})^r}-1.\end{cases}$

Proof. (i) We prove (ii) to (v) as follows, (i) is clear.

(ii) Carrying out an elementary row operation on $[L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ we obtain a matrix with an echelon form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1_{p^{(\frac{(s^2+3s)}{2})r}-2} \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 & -1_{p^{(\frac{(s^2+3s)}{2})r}-1} \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & -1_{p^{(\frac{(s^2+3s)}{2})r}} \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & -1_{p^{(\frac{(s^2+3s+2)}{2})r}-3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & -1_{p^{(\frac{(s^2+3s+2)}{2})r}-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This results to $p^{(\frac{(s^2+3s+2)}{2})r}-2$ non-zero rows in $[L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$, hence the rank.

(iii) Since $\mid Z(R)^* \mid = p^{(\frac{(s^2+3s+2)}{2})r}-1$, each $v_i \in Ann(Z(R)^*)$ has degree $p^{(\frac{(s^2+3s+2)}{2})r}-2$ and $\mid Ann(Z(R)^*) \mid = p^{(\frac{(s^2+3s)}{2})r}-1$. Therefore any $v_j \notin Ann(Z(R)^*)$ is of degree $p^{(\frac{(s^2+3s)}{2})r}-1$ because every such $v_j \notin Ann(Z(R)^*)$ is only adjacent to $v_i \in Ann(Z(R)^*)$.
Partitioning $V \in \Gamma(R)$ into disjoint subsets V_1 and V_2 such that
$V_1 = \{v_j | v_j \notin Ann(Z(R)^*)\}$ and $V_2 = \{v_i | v_i \in Ann(Z(R)^*)\}$.
Therefore, $\mid V_2 \mid = p^{(\frac{(s^2+3s)}{2})r}-1$ and $\mid V_1 \mid = p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}$.

Since the trace, $Tr([L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = \sum_{i=1}^{p^{(\frac{(s^2+3s+2)}{2})r}-1} d_{ii}$, and every d_{ii} is an entry of the diagonal matrix $[D]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ whose diagonal are entries of the degrees of $v_i \in V(\Gamma(R))$ thus

$$Tr([L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = (p^{(\frac{(s^2+3s+2)}{2})r}-2)(p^{(\frac{(s^2+3s)}{2})r}-1) + (p^{(\frac{(s^2+3s)}{2})r}-1)(p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}).$$

Upon expansion and simplification of this equation, we obtain

$$p^{(\frac{2(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 2p^{(\frac{(s^2+3s)}{2})r} + 2 + p^{(\frac{2(s^2+3s)}{2})r} - p^{(2\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} =$$
$$2p^{(\frac{2(s^2+3s)}{2})r} - 2p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r} - 2p^{2(\frac{(s^2+3s)}{2})r} + 2 = 2p^{(\frac{2(s^2+3s)}{2})r} - 2p^{2(\frac{(s^2+3s)}{2})r} - 2p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r} + 2.$$

(iv) Simplifying $\mid [L]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}} \mid = \sum_{i,j=1}^{p^{(\frac{(s^2+3s+2)}{2})^r-1}} a_{ij}(-1)^{i+j} \mid l_{ij} \mid$ on the Laplacian matrix where l_{ij} are minors to $[L]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}}$ and a_{ij} are the row or column elements from the row or the column of operation, we then establish the singularity of $[L]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}}$.
(v) Solving $\mid (\lambda I_{p^{(\frac{(s^2+3s+2)}{2})^r-1}} - [L]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}}) \mid = 0$ gives the characteristic polynomial equation of the form;

$$-\lambda((-p^{(\frac{(s^2+3s+2)}{2})^r} - 1) + \lambda)^{p^{(\frac{(s^2+3s)}{2})^r-1}}(-(p^{(\frac{(s^2+3s)}{2})^r} - 1) + \lambda)^{p^{(\frac{(s^2+3s)}{2})^r-1}}) = 0.$$

On solving each factor, we obtain $-\lambda = 0 \implies \lambda = 0$. For the factor $(-(p^{(\frac{(s^2+3s+2)}{2})^r} - 1) + \lambda)^{p^{(\frac{(s^2+3s)}{2})^r-1}} = 0$, we have that $\lambda = (p^{(\frac{(s^2+3s+2)}{2})^r} - 1)$ of multiplicity $p^{(\frac{(s^2+3s)}{2})^r} - 1$. Finally, $(-(p^{(\frac{(s^2+3s)}{2})^r} - 1) + \lambda) = 0 \implies \lambda = p^{(\frac{(s^2+3s)}{2})^r} - 1$, this establishes (v).

□

Proposition 4.5. *Consider $[d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}}$, the distance matrix associated with $\Gamma(R)$ for the ring in Construction II such that $pu_i = 0$. Then for $r \in \mathbb{Z}^+, p,$ prime and s fixed,*

(i) $[d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}}$ *is a singular matrix.*

(ii) $Tr([d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}}) = 0.$

(iii) $rank([d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}}) = p^{(\frac{(s^2+3s+2)}{2})^r} - 2.$

(iv) Eigenvalues $\lambda[d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}} = \begin{cases} 0, & \\ p^{(\frac{(s^2+3s+2)}{2})^r}, & \\ -1, & \text{of multiplicity } p^r, \\ -p^r, & \text{of multiplicity } p^r+1. \end{cases}$

Proof. (i) The proofs for (i), (ii) and (iii) are clear.
(iv) Solving the equation $\mid \lambda I - [d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})^r-1}} \mid = 0$ results to the characteristic equation $-(-p^{(\frac{(s^2+3s+2)}{2})^r} + \lambda)\lambda(1+\lambda)^{p^r}(p^r+\lambda)^{p^r+1} = 0$. We obtain the eigenvalues by solving every factor of the equation as follows: Clearly, $\lambda = 0$. Further, $-(-p^{(\frac{(s^2+3s+2)}{2})^r} + \lambda) = 0 \implies \lambda = p^{(\frac{(s^2+3s+2)}{2})^r}$. For $(1+\lambda)^{p^r} = 0, \lambda = -1$ of multiplicity p^r. Finally, $(p^r+\lambda)^{p^r+1} = 0 \implies \lambda = -p^r$ of multiplicity p^r+1.

□

Case II: when $pu_i \neq 0$

Proposition 4.6. *Consider* R *from Construction II such that* $pu_i \neq 0$. *Then the zero divisors set* $Z(R)$ *satisfy the following properties;*

(i) $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$.

(ii) $(Z(R))^2 = pR' \oplus \sum_{i,j=1}^{s} R'u_iu_j$.

(iii) $(Z(R))^3 = (0)$.

Proposition 4.7. *Let* R *be the 3-radical zero completely primary finite ring of characteristic* p^2 *in Construction II such that* $pu_i \neq 0$ *and* $\Gamma(R)$ *be its zero divisor graph. Then for* $r \in \mathbb{Z}^+, p$ *prime and* s *fixed,*

(i) $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1$.

(ii) $\Gamma(R)$ *is incomplete.*

(iii) $diam(\Gamma(R)) = 2$.

(iv) $\delta(\Gamma(R)) = p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 1$.

(v) $girth(\Gamma(R)) = 3$.

Proof. (i). Given that $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$ and since $\mid R' \mid = p^{2r}, \mid R'u_i \mid = p^{2r}, \mid R'u_iu_j \mid = p^r$ for $i, j = 1, 2, \cdots, s$ it implies that $\mid Z(R) \mid = p^{(\frac{(s^2+5s+2)}{2})r}$. Moreover, since $\mid Z(R)^* \mid = \mid Z(R) - \{0\} \mid$ it means that $\mid V(\Gamma(R)) \mid = \mid Z(R)^* \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1$.
(ii). Since $(Z(R))^2 \neq (0)$, it follows that not all vertices $v_i, v_j \in V(\Gamma(R))$ are connected by an edge. This expalins incompleteness of $\Gamma(R)$.
(iii). There exist non adjacent vertices $v_i, v_k \in V(\Gamma(R))$ due to (ii) so that for some vertex $v_j \in Ann(Z(R)) = (Z(R))^2$, the longest path of the graph is $v_i - v_j - v_k$. Which establishes (iii).
(iv). As established in (i), $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1$. Clearly $\mid Ann(Z(R)) - \{0\} \mid = \mid Ann(Z(R))^* \mid = p^{(\frac{(s^2+3s+2)}{2})r} - 1$. The minimum degree from the graph can be obtained by computing the order $\mid V(\Gamma(R)) \setminus Ann(Z(R))^* \mid = (p^{(\frac{(s^2+5s+2)}{2})r} - 1) - (p^{(\frac{(s^2+3s+2)}{2})r} - 1) = p^{(\frac{(s^2+5s+2)}{2})r} - 1 - p^{(\frac{(s^2+3s+2)}{2})r} + 1 = p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}$. For avoidance of self loop, we have that $\delta(\Gamma(R)) =$

$p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 1.$
(v) Follows from Proof (v) in Proposition 3.2.

□

Proposition 4.8. *Consider* $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ *and* $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ *to be respectively the adjacency and Laplacian matrices for* $\Gamma(R)$ *for the ring in Construction II such that* $pu_i \neq 0,\ r \in \mathbb{Z}^+,\ p$ *prime and* s *fixed. Then*

(i) $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ *and* $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ *are both symmetric.*

(ii) $rank([A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = p^{(\frac{(s^2+3s+2)}{2})r}$ *and* $rank([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = p^{(\frac{(s^2+5s+2)}{2})r} - 2.$

(iii) $Det([A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = Det([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = 0.$

(iv) $Tr([A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = 0,$
$Tr([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = 2p^{(\frac{2(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{2(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} + 2.$

(v) The eigenvalues $\lambda[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$

$$= \begin{cases} -1, & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 2, \\ 0, & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 1, \\ (p^r+1) \pm \sqrt{\Omega} & . \end{cases}$$

where $\Omega = (p^r+1)^2 + p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(h+2)r}.$

(vi) The eigenvalues of $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$

$$= \begin{cases} 0, & \\ p^{(\frac{(s^2+5s+2)}{2})r} - 1, & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 1, \\ (p^{(\frac{(s^2+3s+2)}{2})r} - 1), & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 1. \end{cases}$$

Proof. Proofs for Properties (i) to (iii) of $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ and $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ are clear.
We proceed to provide proof for (iv), (v) and (vi) as follows.
(iv). For the adjacency matrix, $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$, the result is clear since diagonal entries are all $0'$s.
Since the order is $p^{(\frac{(s^2+5s+2)}{2})r}-1$, it follows that $\sum_{i=1}^{p^{(\frac{(s^2+5s+2)}{2})r}-1} a_{ii}=0$ where a_{ii} are the diagonal elements of the matrix $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$.
We show that $Tr([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = 2p^{(\frac{2(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} + 2.$
Since $\mid Z(R)^* \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1 = \mid V(\Gamma(R)) \mid$, it is established that $\mid Ann(Z(R)^*) \mid = p^{(\frac{(s^2+3s+2)}{2})r} - 1$ and any $v_i \notin Ann(Z(R)^*)$ is of degree $p^{(\frac{(s^2+3s+2)}{2})r} - 1$ since v_i is only adjacent to the vertices in $Ann(Z(R)^*)$. In the same manner, each v_j in the set $Ann(Z(R)^*)$ is connected by an edge with $v_i \in V(\Gamma(R))$. Therefore, $deg(v_j) = p^{(\frac{(s^2+5s+2)}{2})r} - 2$ for avoidance of self loop.
Let the partitions of the vertex set in $\Gamma(R)$ be V_1 and V_2 such that
$V_1 = \{v_i \in Z(R)^* | v_i \notin Ann(Z(R)^*)\}$ and $V_2 = \{v_j \in Z(R)^* | v_j \in Ann(Z(R)^*)\} \Longrightarrow$
$\mid V_1 \mid = p^{(\frac{(s^2+3s+2)}{2})r} - 1$ and $\mid V_2 \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1 - (p^{(\frac{(s^2+3s+2)}{2})r} - 1) = p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}$.
Since the trace of $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ is the sum of the diagonal entries of the degree matrix $[D]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$, that is $Tr([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = \sum_{i=1}^{p^{(\frac{(s^2+5s+2)}{2})r}-1} d_{ii}$ which is equivalent to sum of degrees of the vertices in $\Gamma(R)$ where d_{ii} are the diagonal entries of $[D]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$. We have that

$$(p^{(\frac{(s^2+3s+2)}{2})r} - 1)(p^{(\frac{(s^2+5s+2)}{2})r} - 2) + (p^{(\frac{(s^2+3s+2)}{2})r} - 1)(p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}) =$$

$p^{(\frac{2(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} + 2 + p^{(\frac{2(s^2+5s+2)}{2})r} - p^{(\frac{2(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}$
which simplifies to
$2p^{(\frac{2(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{2(s^2+3s+2)}{2})r} + 2$. Hence the trace of $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$.

(v). Simplifying the equation $\mid \lambda I_{p^{(\frac{(s^2+5s+2)}{2})r}-1} - [A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1} \mid = 0$ results to the characteristic polynomial equation

$$-\lambda^{p^{(\frac{(s^2+3s+2)}{2})r}-1}(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2}(\lambda^2 - 2(p^r+1)\lambda - (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})) = 0.$$

From the factorization components, $-\lambda^{p^{(\frac{(s^2+3s+2)}{2})r-1}} = 0 \Rightarrow \lambda = 0$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 1$.
Similarly, $(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r-2}} = 0 \Rightarrow \lambda = -1$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 2$.
For the quadratic part, $\lambda^2 - 2(p^r+1)\lambda - (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}) = 0 \Rightarrow$

$$\lambda = \frac{2(p^r+1) \pm \sqrt{4(p^r+1)^2 + 4p^{(\frac{(s^2+5s+2)}{2})r} + 4p^{(\frac{(s^2+5s+2)}{2})r} + 4p^{(\frac{(s^2+3s+2)}{2})r}}}{2}$$
$$= \frac{2(p^r+1) \pm \sqrt{4((p^r+1)^2 + p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})}}{2}$$
$$= (p^r+1) \pm \sqrt{(p^r+1)^2 + p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}}$$
$$= (p^r+1) \pm \sqrt{(p^r+1)^2 + p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+5s+4)}{2})r}}.$$

(vi). From the characteristic equation $\mid \lambda I_{p^{(\frac{(s^2+5s+2)}{2})r}-1} - [L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1} \mid = 0$, we obtain $-((-(p^{(\frac{(s^2+5s+2)}{2})r} - 1) + \lambda)^{p^{(\frac{(s^2+3s+2)}{2})r-1}}(-(p^{(\frac{(s^2+3s+2)}{2})r} - 1) + \lambda)^{p^{hr}-1})\lambda = 0$. Solving λ for each factor results to $(-(p^{(\frac{(s^2+5s+2)}{2})r} - 1) + \lambda)^{p^{(\frac{(s^2+3s+2)}{2})r-1}} = 0 \Longrightarrow \lambda = p^{(\frac{(s^2+5s+2)}{2})r} - 1$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 1$. Further, $(-(p^{(\frac{(s^2+3s+2)}{2})r} - 1) + \lambda)^{p^{(\frac{(s^2+3s+2)}{2})r-1}} = 0 \Longrightarrow \lambda = p^{(\frac{(s^2+3s+2)}{2})r} - 1$ with a multiplicity of $p^{(\frac{(s^2+3s+2)}{2})r} - 1$, and finally, $\lambda = 0$. Hence the eigenvalues of $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$.

□

Proposition 4.9. *Given $[d_{ij}]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$, the distance matrix of associated with $\Gamma(R)$ of the ring in Construction II. The point spectrum, $\sigma_{point}([d_{ij}]_{p^{(\frac{(s^2+5s+2)}{2})r}-1})$ is described by the following eigenvalues:*

$$\lambda = \begin{cases} -1, & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 2; \\ -p^r, & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 1; \\ (p^{(\frac{(s^2+3s+2)}{2})r} + 2) \pm \sqrt{p^{(\frac{2(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}}, & \end{cases}$$

Proof. Simplifying the equation $\mid \lambda I - [d_{ij}]_{p^{(\frac{(s^2+5s+2)}{2})r}-1} \mid = 0$ results to the polynomial equation of the form
$-(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r-2}}(p^r+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r-1}}(\lambda^2 - (2p^{(\frac{(s^2+3s+2)}{2})r})\lambda) + (3p^{(\frac{(s^2+3s+2)}{2})r} + 4)) = 0.$

Finding the values of λ in each factor yields $-(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2} = 0 \Longrightarrow \lambda = -1$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 2$. Further, $(p^r+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-1} = 0 \Longrightarrow \lambda = -p^r$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 1$.
For the quadratic part $\lambda^2 - (2p^{(\frac{(s^2+3s+2)}{2})r})\lambda + (3p^{(\frac{(s^2+3s+2)}{2})r} + 4) = 0$, we obtain

$$\lambda = \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm \sqrt{(2p^{(\frac{(s^2+3s+2)}{2})r}+4)^2 - 4(3p^{(\frac{(s^2+3s+2)}{2})r}+4)}}{2}$$

$$= \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm \sqrt{(4p^{(\frac{2(s^2+3s+2)}{2})r} + 8p^{(\frac{(s^2+3s+2)}{2})r} + 8p^{(\frac{(s^2+3s+2)}{2})r} + 16 - 12p^{(\frac{(s^2+3s+2)}{2})r} - 16)}}{2}$$

$$= \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm \sqrt{4p^{(\frac{2(s^2+3s+2)}{2})r} + 16p^{(\frac{(s^2+3s+2)}{2})r} - 12p^{(\frac{(s^2+3s+2)}{2})r}}}{2}$$

$$= \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm \sqrt{4p^{(\frac{2(s^2+3s+2)}{2})r} + 4p^{(\frac{(s^2+3s+2)}{2})r}}}{2}$$

$$= \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm 2\sqrt{p^{(\frac{2(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}}}{2}$$

$$= (p^{(\frac{(s^2+3s+2)}{2})r}+2) \pm \sqrt{p^{(\frac{2(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}}.$$

□

5 The 3-Radical Zero Finite Completely Primary Rings of Characteristic p^3

5.1 Construction III

Let $R' = GR(p^{3r}, p^3)$ be a Galois ring of order p^{3r} and characteristic p^3. Let U and V be finitely generated R'-modules with the generating sets $\{u_1, u_2, \cdots, u_s\}$ and $\{v_1, v_2, \cdots, v_t\}$ respectively such that s and t are the number of elements in the generating sets. Suppose $t = \frac{s(s+1)}{2}$ for a fixed s, $R = R' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$ is an additive abelian group. Define multiplication on R by

$$(x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j)(y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j) =$$

$$x_\circ y_\circ + \sum_{i=1}^{s}((x_\circ + pR')y_i + x_i(y_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(x_\circ y_j + x_j(y_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} a_{ij}x_i(y_j)^{\sigma_i})u_iu_j.$$

The multiplication given turns R into a commutative ring with identity $(1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0})$ if $\sigma_i = id_{\mathbb{F}}$. From this multiplication, the set $Z(R)$ of zero divisors satisfy the following properties;

(i) $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i + \sum_{i,j=1}^{s} R'u_iu_j$,

(ii) $(Z(R))^2 = p^2R' \oplus \sum_{i,j=1}^{s} R'u_iu_j$,

(iii) $(Z(R))^3 = (0)$.

For the rings considered in this section, $\sigma_i = id_{\mathbb{F}}$.

5.2 The Graphs $\Gamma(R)$ and Matrices from Classes of Rings in Construction III

Proposition 5.1. *Given R, the ring of Construction III and $\Gamma(R)$ be the associated zero divisor graph. Then for any prime integer $p, r \in \mathbb{Z}^+$ and s-fixed.*

(i) $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+5s+4)}{2})r} - 1$.

(ii) $\Delta(\Gamma(R)) = p^{(\frac{(s^2+5s+4)}{2})r} - 2$ *and* $\delta(\Gamma(R)) = p^{(\frac{(s^2+5s)}{2})r}$.

(iii) $\Gamma(R)$ *is incomplete.*

(iv) $diam(\Gamma(R)) = 2$.

(v) $girth(\Gamma(R)) = 3$.

Proof. (i) Since $char(R) = p^3$, $\mid R' \mid= p^{3r}$ and $\mid pR' \mid= p^{2r}$. Consider $pu_i = 0$ for $i = 1, 2, \cdots, s$ and $\mid R'u_iu_j \mid= p^r, i, j = 1, 2, \cdots, s$ we have that $\mid Z(R) \mid= p^{(\frac{(s^2+5s+4)}{2})r}$ and $\mid Z(R) \setminus \{0\} \mid=$ $\mid (Z(R))^* \mid=\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+5s+4)}{2})r} - 1$.
(ii) Let $\gamma_1, \gamma_2, \cdots, \gamma_r \in R'$ with $\gamma_1 = 1$ such that $\overline{\gamma}_1, \cdots, \overline{\gamma}_r \in R'$ forms a basis for R' over its prime subfield R'/pR'. From the multiplication defined on R,
$Ann(Z(R)) = \{p^2r_\circ + \sum_{i=1}^{s} a_i\gamma_iu_i + \sum_{i=1}^{s} b_i\gamma_iu_iu_j | a_i, b_i \in R', a_i + b_i \cong 0(mod\ p)\}$. With the fact that $\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+5s+4)}{2})r} - 1$, any vertex $v_i \in Ann(Z(R))^*$ is of degree $p^{(\frac{(s^2+5s+4)}{2})r} - 2$ due to avoidance of self loop. Hence the maximum degree $\Delta(\Gamma(R))$.
Partitioning $V(\Gamma(R))$ into disjoint subsets V_1 and V_2 such that $V_1 = \{v_i | v_i \in Ann(Z(R))^*\}$ and $V_2 = \{v_j | v_j \notin Ann(Z(R))^*\}$, $\mid V_1 \mid= p^{(\frac{(s^2+5s)}{2})r}$. This implies that the vertices of minimum degree are only adjacent to $v_i \in Ann(Z(R))^*$ and since $\mid V_1 \mid= p^{(\frac{(s^2+5s)}{2})r}$, $\delta(\Gamma(R)) = p^{(\frac{(s^2+5s)}{2})r}$.
(iii) to (v) are clear. □

The results in the sequel describe the algebraic properties of the matrices associated with $\Gamma(R)$ of the ring in Construction III.

Proposition 5.2. *Given* $[A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}$ *and* $[L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}$, *the adjacency and Laplacian matrices of* $\Gamma(R)$ *respectively for the ring in Construction III. Then for a prime integer* $p, r \in \mathbb{Z}^+$ *and* s *fixed,*

(i) $Det([A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}) = Det([L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}) = 0$.

(ii) $rank([A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}) = p^{(\frac{(s^2+3s+2)}{2})r}+2$ *and* $rank([L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}) = p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s+2)}{2})r}+1$.

(iii) Eigenvalues $\lambda[A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1} =$

$$\begin{cases} 0, & \textit{of multiplicity } p^{(\frac{(s^2+5s+2)}{2})r} + p^r + 1; \\ -1, & \textit{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 2; \\ -p^{(\frac{(s^2+3s+2)}{2})r}, & \textit{of multiplicity } p^r; \\ p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1 \pm \rho. \end{cases}$$

Where

$$\rho = \sqrt{(p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1)^2 - (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})}.$$

(iv) Eigenvalues $\lambda[L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1} =$

$$\begin{cases} 0, & \textit{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 2; \\ 1, & \textit{of multiplicity } p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} + 1; \\ 1 - p^{(\frac{(s^2+3s+2)}{2})r}, & \textit{of multiplicity } p^r; \\ p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} \pm \epsilon. \end{cases}$$

Where

$$\epsilon = \sqrt{(p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r})^2 - (p^{(\frac{(s^2+5s+4)}{2})r} + 2p^{(\frac{(s^2+5s+2)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} - 1)}.$$

Proof. We provide proofs for (iii) and (iv) since the steps for proofs in (i) and (ii) are clear.

(iii) Solving the determinant $\mid \lambda I - [A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1} \mid = 0$, we obtain the characteristic polynomial equation of the form
$-\lambda^{(p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r}+1)}(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2}(p^{(\frac{(s^2+3s+2)}{2})r}+\lambda)^{p^r}(\lambda^2 - 2(p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1)\lambda + (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})) = 0.$
Finding the value of λ from each factor in the above equation results to
$-\lambda^{p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r}+1} = 0 \Longrightarrow \lambda = 0$ of multiplicity $p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} + 1.$
$(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2} = 0 \Longrightarrow \lambda = -1$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 2$ and the factor $(p^{(\frac{(s^2+3s+2)}{2})r} + \lambda)^{p^r} =$

$0 \Longrightarrow \lambda = -p^{(\frac{(s^2+3s+2)}{2})r}$ of multiplicity p^r.
The quadratic part $\lambda^2 - 2(p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1)\lambda + (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}) = 0$ can be solved as

$$\lambda = \frac{2(p^{(\frac{(s^2+3s+2)}{2})r}+p^r+1)\pm\sqrt{4(p^{(\frac{(s^2+3s+2)}{2})r}+p^r+1)^2-4(p^{(\frac{(s^2+5s+4)}{2})r}+p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s+2)}{2})r})}}{2}.$$

$$= \frac{2(p^{(\frac{(s^2+3s+2)}{2})r}+p^r+1)\pm\sqrt{4((p^{(\frac{(s^2+3s+2)}{2})r}+p^r+1)^2-(p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s+2)}{2})r}))}}{2}.$$

Simplifying this equation yields

$$\lambda = (p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1) \pm \sqrt{(p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1)^2 - (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})}.$$

(iv) Similarly, we can provide proof for eigenvalues of $[L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}$ by solving the equation $| \lambda I - [L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1} |= 0$ which results to the polynomial equation
$-((-1+\lambda)^{p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r}+1}\lambda^{p^{(\frac{(s^2+3s+2)}{2})r}-2}((p^{(\frac{(s^2+3s+2)}{2})r}-1)+\lambda)^p(\lambda^2-2(p^{(\frac{(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r})\lambda+ (p^{(\frac{(s^2+5s+4)}{2})r} + 2p^{(\frac{(s^2+5s+2)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} - 1)) = 0.$
Solving the equation leads to $(-1+\lambda)^{p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r}+1} = 0 \Longrightarrow \lambda = 1$ of multiplicity $p^{(\frac{(s^2+5s+2)}{2})r}+ p^{(\frac{(s^2+3s)}{2})r} + 1, \lambda^{p^{(\frac{(s^2+3s+2)}{2})r}-2} = 0 \Longrightarrow \lambda = 0$ of algebraic multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 2$. Similarly, $((p^{(\frac{(s^2+3s+2)}{2})r} - 1) + \lambda)^{p^r} = 0 \Longrightarrow \lambda = 1 - p^{(\frac{(s^2+3s+2)}{2})r}$ of multiplicity p^r. The quadratic part
$\lambda^2 - 2(p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r})\lambda + (p^{(\frac{(s^2+5s+4)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} - 1) = 0$ can be solved as follows;

$$\lambda = \frac{2(p^{(\frac{(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r})\pm\sqrt{4(p^{(\frac{(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r})^2-4(p^{(\frac{(s^2+5s+4)}{2})r}+2p^{(\frac{(s^2+5s+4)}{2})r}+2p^{(\frac{(s^2+3s+2)}{2})r}-1)}}{2}$$

$$= \frac{2(p^{(\frac{(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r})\pm2\sqrt{(p^{(\frac{(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r})^2-(p^{(\frac{(s^2+5s+4)}{2})r}+2p^{(\frac{(s^2+5s+2)}{2})r}+2p^{(\frac{(s^2+3s+2)}{2})r}-1)}}{2}$$

$$= (p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r}) \pm \epsilon.$$

where $\epsilon = \sqrt{(p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r})^2 - (p^{(\frac{(s^2+5s+4)}{2})r} + 2p^{(\frac{(s^2+5s+2)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} - 1)}.$ □

6 Conclusion

This research has succeeded in representing the zero divisor graphs $\Gamma(R)$ from classes of 3-radical zero completely primary finite rings using the Adjacency, Laplacian and Distance Matrices which have been analyzed for some of their algebraic properties.

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