On The Sum of Three Square Formula

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ABSTRACT

Let n, x, y, z be any given integers. The study of n for which $n = x^2 + y^2 + z^2$ is a very long standing problem. Recent survey of sizeable literature shows that many researchers have made some progress to come up with algorithms of decomposing integers into sums of three squares. On the other hand, available results on integer representation as sums of three square is still very minimal. If $a, b, c, d, k, m, n, u, v$ and w are any non-negative integers, this study determines the sum of three square formula of the form $abcd + ka^2 + ma + n = u^2 + v^2 + w^2$ and establishes its applications to various cases.

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1 Introduction

The study of integer n for which n is a sum of three squares is still an open area of research. Most of the research done on n has put more focus on developing an algorithm for integer sums of three squares (see. [9]) with very minimal focus given to general formulas for integer sums of three squares. Perhaps, this is because of the fact that the study of n for which $n = x^2 + y^2 + z^2$ is not an easy task. Some contribution on sums of three squares can be mentioned. The determination of integer sums of three squares was invigorated by research of Legendre in [7], where he stated that a natural number can be represented as the sum of three squares of integers $n = x^2 + y^2 + z^2$ if and only if n is not of the form $n = 4^m(8k + 7)$ where m and k are non-negative integers. The proof to this assertion was provided by

legendre himself in 1798. Arenas and Bayer [1], did considerable work on arithemetic behavior of sums of three squares by computing the solvability of $n = x^2 + y^2 + z^2$ where x, y and z are relatively prime to n via Vila's theorem. The study of integer representation as a sum of three squares was also attempted by Hirschhorn and Sellers in [4]. In [3] Gauss proved his Eureka theorem that every positive integer n is the sum of 3 triangular numbers; this is equivalent to the fact that $8n + 3$ is a sum of three squares. Some recent surveys on sums of three squares, such as those by Deshouillers and Luca's investigation in [2], titled "How often is n! a sum of three squares?", provides insights into the frequency of $n!$ being a sum of three squares, adding another layer to the multifaceted exploration of sums of squares in number theory. Additionally, randomized algorithms in number theory, as explored by Rabin and Shallit in [9],have provided valuable computational methods for solving problems related to sums of three squares. Rob in [10] contributed to the field with work on factorials and Legendre's three-square theorem, while Yingchun in [11] focused on Gauss's Three Square Theorem almost involving primes. Some recent examination on sums of squares and sums of cube can be attributed to [5, 6, 8, 11]. This research therefore improves the existing results concerning sums of squares by introducing and developing sums of three square formula given by $abcd + ka^2 + ma + n = u^2 + v^2 + w^2$ where $a, b, c, d, k, m, n, u, v$ and w are all positive integers.

2   Main Results

In the sequel we present some identities for sums of three squares. The following holds throughout this study $d > c > b > a$.

Proposition 2.1. $abcd + 2a^2 + 6a + 21 = \begin{cases} (a+1)^2 + (a+2)^2 + (a^2 + 6a + 4)^2 \cdots\cdots (I), \\ (b-1)^2 + b^2 + (b^2 + 2b - 4)^2 \cdots\cdots\cdots\cdots (II), \\ (c-3)^2 + (c-2)^2 + (c^2 - 2c - 4)^2 \cdots\cdots (III), \\ (d-5)^2 + (d-4)^2 + (d^2 - 6d + 4)^2 \cdots\cdots (IV). \end{cases}$

has solution in integers if a, b, c *and* d *are consecutive integers of same parity.*

Proof. Suppose a, b, c and d are consecutive integers of same parity. Then, $b = a+2, c = a+4, d = a+6$. So, $abcd + 2a^2 + 6a + 21 = a(a+2)(a+4)(a+6) + 2a^2 + 6a + 21 = (a^2 + 2a)(a+4)(a+6) + 2a^2 + 6a + 21 = [a^2(a+4) + 2a(a+4)](a+6)2a^2 + 6a + 21 = (a^3 + 4a^2 + 2a^2 + 8a)(a+6) + 2a^2 + 6a + 21 = a(a^3 + 4a^2 + 2a^2 + 8a) + 6(a^3 + 4a^2 + 2a^2 + 8a) + 2a^2 + 6a + 21 = a^4 + 6a^3 + 8a^2 + 6a^3 + 36a^2 + 48a + 2a^2 + 6a + 21 = a^4 + 12a^3 + 46a^2 + 54a + 21 \cdots (2.1)$.
Decomposing equation (2.1) into sums of three squares, we have $a^4 + 12a^3 + 46a^2 + 54a + 21 = a^4 + 12a^3 + 44a^2 + 48a + 16 + a^2 + 2a + 1 + a^2 + 4a + 4 = (a^2 + 6a + 4)^2 + (a+1)^2 + (a+2)^2$ proving the results for case (I).
Next, the proofs of cases $(II), (III)$ and (IV) follow easily from case (I) with some slight modification.

$\square$

Proposition 2.2. $abcd + 2a^2 + 14a + 41 = \begin{cases} (a+3)^2 + (a+4)^2 + (a^2+6a+4)^2 \cdots (I), \\ (b-1)^2 + b^2 + (b^2+2b-4)^2 \cdots\cdots\cdots (II), \\ (c-3)^2 + (c-2)^2 + (c^2-2c-4)^2 \cdots (III), \\ (d-5)^2 + (d-4)^2 + (d^2-6d+4)^2 \cdots (IV). \end{cases}$

has solution in integers if a, b, c and d are consecutive even integers of same parity.

Proof. Suppose a, b, c and d are consecutive integers of same parity. Then, $b = a+2, c = a+4, d = a+6$. So, $abcd + 2a^2 + 14a + 41 = a(a+2)(a+4)(a+6) + 2a^2 + 14a + 41 = (a^2+2a)(a+4)(a+6) + 2a^2 + 14a + 41 = [a^2(a+4) + 2a(a+4)](a+6)2a^2 + 14a + 41 = (a^3+4a^2+2a^2+8a)(a+6) + 2a^2 + 14a + 41 = a(a^3+4a^2+2a^2+8a) + 6(a^3+4a^2+2a^2+8a) + 2a^2 + 14a + 41 = a^4 + 6a^3 + 8a^2 + 6a^3 + 36a^2 + 48a + 2a^2 + 14a + 41 = a^4 + 12a^3 + 46a^2 + 62a + 41 \cdots (2.2)$.
Splitting equation (2.2) into sums of three squares, we have $a^4 + 12a^3 + 46a^2 + 62a + 41 = a^4 + 12a^3 + 44a^2 + 48a + 16 + a^2 + 6a + 9 + a^2 + 8a + 16 = (a^2+6a+4)^2 + (a+3)^2 + (a+4)^2$ establishing the results for case (I).
The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes. $\square$

Proposition 2.3. $abcd + 2a^2 + 14a + 45 = \begin{cases} (a+2)^2 + (a+5)^2 + (a^2+6a+4)^2 \cdots (I), \\ (b-1)^2 + b^2 + (b^2+2b-4)^2 \cdots\cdots\cdots (II), \\ (c-3)^2 + (c-2)^2 + (c^2-2c-4)^2 \cdots (III), \\ (d-5)^2 + (d-4)^2 + (d^2-6d+4)^2 \cdots (IV). \end{cases}$

has solution in integers if a, b, c and d are consecutive integers of same parity.

Proof. Suppose a, b, c and d are consecutive integers of same parity. Then, $b = a+2, c = a+4, d = a+6$. So, $abcd + 2a^2 + 14a + 45 = a(a+2)(a+4)(a+6) + 2a^2 + 14a + 45 = (a^2+2a)(a+4)(a+6) + 2a^2 + 14a + 45 = [a^2(a+4) + 2a(a+4)](a+6)2a^2 + 14a + 45 = (a^3+4a^2+2a^2+8a)(a+6) + 2a^2 + 14a + 45 = a(a^3+4a^2+2a^2+8a) + 6(a^3+4a^2+2a^2+8a) + 2a^2 + 14a + 45 = a^4 + 6a^3 + 8a^2 + 6a^3 + 36a^2 + 48a + 2a^2 + 14a + 45 = a^4 + 12a^3 + 46a^2 + 62a + 45 \cdots (2.3)$.
Decomposing equation (2.3) into sums of three squares, we have $a^4 + 12a^3 + 46a^2 + 62a + 45 = a^4 + 12a^3 + 44a^2 + 48a + 16 + a^2 + 4a + 4 + a^2 + 10a + 25 = (a^2+6a+4)^2 + (a+2)^2 + (a+5)^2$ giving the results for case (I).
The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes. $\square$

Proposition 2.4. $abcd + 2a^2 + 10a + 33 = \begin{cases} (a+1)^2 + (a+4)^2 + (a^2+6a+4)^2 \cdots (I), \\ (b-1)^2 + b^2 + (b^2+2b-4)^2 \cdots\cdots\cdots (II), \\ (c-3)^2 + (c-2)^2 + (c^2-2c-4)^2 \cdots (III), \\ (d-5)^2 + (d-4)^2 + (d^2-6d+4)^2 \cdots (IV). \end{cases}$

has solution in integers if a, b, c and d are consecutive integers of same parity.

Proof. Suppose a, b, c and d are consecutive integers of same parity. Then, $b = a+2, c = a+4, d = a+6$. So, $abcd + 2a^2 + 10a + 33 = a(a+2)(a+4)(a+6) + 2a^2 + 10a + 33 = (a^2+2a)(a+4)(a+6) + 2a^2 + 10a + 33 = $

$[a^2(a+4)+2a(a+4)](a+6)2a^2+10a+33 = (a^3+4a^2+2a^2+8a)(a+6)+2a^2+10a+33 = a(a^3+4a^2+2a^2+8a)+6(a^3+4a^2+2a^2+8a)+2a^2+10a+33 = a^4+6a^3+8a^2+6a^3+36a^2+48a+2a^2+10a+33 = a^4+12a^3+46a^2+58a+33\cdots(2.4)$.

Breaking equation (2.4) into sums of three squares, we have $a^4+12a^3+46a^2+58a+33 = a^4+12a^3+44a^2+48a+16+a^2+2a+1+a^2+8a+16 = (a^2+6a+4)^2+(a+1)^2+(a+4)^2$ proving the results for case (I).

The results for cases $(II),(III)$ and (IV) follow from case (I) with minimal changes. $\square$

Proposition 2.5. $abcd+5a^2+10a+91 = \begin{cases} (2a+1)^2+(a+3)^2+(a^2+9a+9)^2\cdots(I), \\ (2b-5)^2+b^2+(b^2+3b-9)^2\cdots\cdots\cdots(II), \\ (2c-11)^2+(c-3)^2+(c^2-3c-9)^2\cdots(III), \\ (2d-17)^2+(d-6)^2+(d^2-9d-63)^2\cdots(IV). \end{cases}$

has solution in integers if $b = a+3, c = a+6, d = a+9$.

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a+3, c = a+6, d = a+9$. Accordingly, $abcd+5a^2+10a+91 = a(a+3)(a+6)(a+9)+5a^2+10a+91 = [a^2(a+6)+3a(a+6)](a+9)+5a^2+10a+91 = (a^3+6a^2+3a^2+18a)(a+9)+5a^2+10a+91 = (a^3+9a^2+18a)(a+9)+5a^2+10a+91 = a(a^3+9a^2+18a)+9(a^3+9a^2+18a)+5a^2+10a+91 = a^4+9a^3+18a^2+9a^3+81a^2+162a+5a^2+10a+91 = a^4+18a^3+99a^2+162a+5a^2+10a+91 = a^4+18a^3+104a^2+172a+91\cdots(2.5)$.

Decomposing equation (2.5) into sums of three squares we have $a^4+18a^3+104a^2+172a+91 = a^4+18a^3+99a^2+162a+81+4a^2+4a+1+a^2+6a+9 = (2a+1)^2+(a+3)^2+(a^2+9a+9)^2$ proving the results for case (I).

The results for cases $(II),(III)$ and (IV) is similar to case (I). $\square$

Proposition 2.6. $abcd+5a^2+6a+90 = \begin{cases} (a+3)^2+(2a)^2+(a^2+9a+9)^2\cdots\cdots\cdots(I), \\ b^2+(4b^2-24b+36)^2+(b^2+3b-9)^2\cdots(II), \\ (c-3)^2+(2c-12)^2+(c^2-3c-9)^2\cdots(III), \\ (d-6)^2+(2d-18)^2+(d^2-9d-63)^2\cdots(IV). \end{cases}$

has solution in integers if a, b, c, d are integers such that $b = a+3, c = a+6, d = a+9$.

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a+3, c = a+6, d = a+9$. So, $abcd+5a^2+6a+90 = a(a+3)(a+6)(a+9)+5a^2+6a+90 = [a^2(a+6)+3a(a+6)](a+9)+5a^2+6a+90 = (a^3+6a^2+3a^2+18a)(a+9)+5a^2+6a+90 = (a^3+9a^2+18a)(a+9)+5a^2+6a+90 = a(a^3+9a^2+18a)+9(a^3+9a^2+18a)+5a^2+6a+90 = a^4+9a^3+18a^2+9a^3+81a^2+162a+5a^2+6a+90 = a^4+18a^3+99a^2+162a+5a^2+6a+90 = a^4+18a^3+104a^2+168a+90\cdots(2.6)$.

Splitting equation (2.6) into sums of three squares we have $a^4+18a^3+104a^2+168a+90 = a^4+18a^3+99a^2+162a+81+4a^2+a^2+6a+9 = (a+3)^2+(2a)^2+(a^2+9a+9)^2$ proving the results for case (I).

The results for cases $(II),(III)$ and (IV) follow from case (I) with minimal deviation. $\square$

Proposition 2.7. $abcd + 5a^2 + 16a + 118 = \begin{cases} (a+6)^2 + (2a+1)^2 + (a^2+9a+9)^2 \cdots\cdots (I), \\ (b+3)^2 + (2b-5)^2 + (b^2+3b-9)^2 \cdots\cdots (II), \\ c^2 + (2c-11)^2 + (c^2-3c-9)^2 \cdots\cdots\cdots (III), \\ (d-3)^2 + (2d-17)^2 + (d^2-9d-63)^2 \cdots (IV). \end{cases}$

has solution in integers if $b = a+3, c = a+6, d = a+9.$

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a+3, c = a+6, d = a+9$. Thus, $abcd + 5a^2 + 16a + 118 = a(a+3)(a+6)(a+9) + 5a^2 + 16a + 118 = [a^2(a+6) + 3a(a+6)](a+9) + 5a^2 + 16a + 118 = (a^3+6a^2+3a^2+18a)(a+9) + 5a^2 + 16a + 118 = (a^3+9a^2+18a)(a+9) + 5a^2 + 16a + 118 = a(a^3+9a^2+18a) + 9(a^3+9a^2+18a) + 5a^2 + 16a + 118 = a^4+9a^3+18a^2+9a^3+81a^2+162a+5a^2+16a+118 = a^4+18a^3+99a^2+162a+5a^2+16a+118 = a^4 + 18a^3 + 104a^2 + 178a + 118 \cdots (2.7).$

Breaking equation (2.7) into sums of three squares we have $a^4 + 18a^3 + 104a^2 + 178a + 118 = a^4 + 18a^3 + 99a^2 + 162a + 81 + a^2 + 12a + 36 + 4a^2 + 4a + 1 = (a+6)^2 + (2a+1)^2 + (a^2+9a+9)^2$ proving the results for case (I).

The results for cases $(II), (III)$ and (IV) is similar to case (I). $\square$

Proposition 2.8. $abcd + 5a^2 + 34a + 139 = \begin{cases} (a+3)^2 + (2a+7)^2 + (a^2+9a+9)^2 \cdots\cdots (I), \\ b^2 + (2b+1)^2 + (b^2+3b-9)^2 \cdots\cdots\cdots\cdots (II), \\ (c-3)^2 + (2c+1)^2 + (c^2-3c-9)^2 \cdots\cdots (III), \\ (2d-11)^2 + (d-4)^2 + (d^2-9d-63)^2 \cdots (IV). \end{cases}$

has solution in integers if $b = a+3, c = a+6, d = a+9.$

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a+3, c = a+6, d = a+9$. So, $abcd + 5a^2 + 34a + 139 = a(a+3)(a+6)(a+9) + 5a^2 + 34a + 139 = [a^2(a+6) + 3a(a+6)](a+9) + 5a^2 + 34a + 139 = (a^3+6a^2+3a^2+18a)(a+9) + 5a^2 + 34a + 139 = (a^3+9a^2+18a)(a+9) + 5a^2 + 34a + 139 = a(a^3+9a^2+18a) + 9(a^3+9a^2+18a) + 5a^2 + 34a + 139 = a^4+9a^3+18a^2+9a^3+81a^2+162a+5a^2+34a+139 = a^4+18a^3+99a^2+162a+5a^2+34a+139 = a^4 + 18a^3 + 104a^2 + 196a + 139 \cdots (2.8).$

Splitting equation (2.8) into sums of three squares we have $a^4 + 18a^3 + 104a^2 + 196a + 139 = a^4 + 18a^3 + 99a^2 + 162a + 81 + a^2 + 6a + 9 + 4a^2 + 28a + 49 = (a+3)^2 + (2a+7)^2 + (a^2+9a+9)^2$ proving the results for cases (I).

The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes. $\square$

Proposition 2.9. $abcd + 5a^2 + 30a + 126 = \begin{cases} (a+3)^2 + (2a+6)^2 + (a^2+9a+9)^2 \cdots\cdots (I), \\ b^2 + (2b)^2 + (b^2+3b-9)^2 \cdots\cdots\cdots\cdots\cdots (II), \\ (c-3)^2 + (2c-6)^2 + (c^2-3c-9)^2 \cdots\cdots (III), \\ (d-6)^2 + (2d-12)^2 + (d^2-9d-63)^2 \cdots (IV). \end{cases}$

has solution in integers if a, b, c, d *are integers such that* $b = a+3, c = a+6, d = a+9.$

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a + 3, c = a + 6, d = a + 9$. Therefore, $abcd + 5a^2 + 30a + 126 = a(a + 3)(a + 6)(a + 9) + 5a^2 + 30a + 126 = [a^2(a+6) + 3a(a+6)](a+9) + 5a^2 + 30a + 126 = (a^3 + 6a^2 + 3a^2 + 18a)(a+9) + 5a^2 + 30a + 126 = (a^3 + 9a^2 + 18a)(a + 9) + 5a^2 + 30a + 126 = a(a^3 + 9a^2 + 18a) + 9(a^3 + 9a^2 + 18a) + 5a^2 + 30a + 126 = a^4 + 9a^3 + 18a^2 + 9a^3 + 81a^2 + 162a + 5a^2 + 30a + 126 = a^4 + 18a^3 + 99a^2 + 162a + 5a^2 + 10a + 91 = a^4 + 18a^3 + 104a^2 + 192a + 126 \cdots (2.9)$.

Decomposing equation (2.9) into sums of three squares we have $a^4 + 18a^3 + 104a^2 + 192a + 126 = a^4 + 18a^3 + 99a^2 + 162a + 81 + 4a^2 + 24a + 36 + a^2 + 6a + 9 = (a + 3)^2 + (2a + 6)^2 + (a^2 + 9a + 9)^2$ proving the results for case (I).

The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes. □

Proposition 2.10. $abcd + 5a^2 + 36a + 153 = \begin{cases} (a + 6)^2 + (2a + 6)^2 + (a^2 + 9a + 9)^2 \cdots \cdots (I), \\ (b + 3)^2 + (2b)^2 + (b^2 + 3b - 9)^2 \cdots \cdots \cdots (II), \\ c^2 + (2c - 6)^2 + (c^2 - 3c - 9)^2 \cdots \cdots \cdots \cdots (III), \\ (d - 6)^2 + (2d - 12)^2 + (d^2 - 9d - 63)^2 \cdots (IV). \end{cases}$

has solution in integers if a, b, c, d are integers such that $b = a + 3, c = a + 6, d = a + 9$.

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a + 3, c = a + 6, d = a + 9$. So, $abcd + 5a^2 + 36a + 153 = a(a + 3)(a + 6)(a + 9) + 5a^2 + 36a + 153 = [a^2(a + 6) + 3a(a + 6)](a + 9) + 5a^2 + 36a + 153 = (a^3 + 6a^2 + 3a^2 + 18a)(a + 9) + 5a^2 + 36a + 153 = (a^3 + 9a^2 + 18a)(a + 9) + 5a^2 + 36a + 153 = a(a^3 + 9a^2 + 18a) + 9(a^3 + 9a^2 + 18a) + 5a^2 + 36a + 153 = a^4 + 9a^3 + 18a^2 + 9a^3 + 81a^2 + 162a + 5a^2 + 36a + 153 = a^4 + 18a^3 + 99a^2 + 162a + 5a^2 + 36a + 153 = a^4 + 18a^3 + 104a^2 + 198a + 153 \cdots (2.10)$.

Splitting equation (2.10) into sums of three squares we have $a^4 + 18a^3 + 104a^2 + 198a + 153 = a^4 + 18a^3 + 99a^2 + 162a + 81 + a^2 + 12a + 36 + 4a^2 + 24a + 36 = (a + 6)^2 + (2a + 6)^2 + (a^2 + 9a + 9)^2$ proving the results for case (I).

The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes. □

Proposition 2.11. $abcd + 5a^2 + 54a + 234 = \begin{cases} (a + 3)^2 + (2a + 12)^2 + (a^2 + 9a + 9)^2 \cdots (I), \\ b^2 + (2b + 6)^2 + (b^2 + 3b - 9)^2 \cdots \cdots \cdots (II), \\ (c - 3)^2 + (2c)^2 + (c^2 - 3c - 9)^2 \cdots \cdots (III), \\ (d - 6)^2 + (2d - 6)^2 + (d^2 - 9d - 63)^2 \cdots (IV). \end{cases}$

has solution in integers if a, b, c, d are integers such that $b = a + 3, c = a + 6, d = a + 9$.

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a + 3, c = a + 6, d = a + 9$. Accordingly, $abcd + 5a^2 + 54a + 234 = a(a + 3)(a + 6)(a + 9) + 5a^2 + 54a + 234 = [a^2(a+6) + 3a(a+6)](a+9) + 5a^2 + 54a + 234 = (a^3 + 6a^2 + 3a^2 + 18a)(a+9) + 5a^2 + 54a + 234 = (a^3 + 9a^2 + 18a)(a + 9) + 5a^2 + 54a + 234 = a(a^3 + 9a^2 + 18a) + 9(a^3 + 9a^2 + 18a) + 5a^2 + 54a + 234 = a^4 + 9a^3 + 18a^2 + 9a^3 + 81a^2 + 162a + 5a^2 + 54a + 234 = a^4 + 18a^3 + 99a^2 + 162a + 5a^2 + 54a + 234 = a^4 + 18a^3 + 104a^2 + 216a + 234 \cdots (2.11)$.

Breaking equation (2.11) into sums of three squares we have $a^4 + 18a^3 + 104a^2 + 216a + 234 = a^4 +$

$18a^3 + 99a^2 + 162a + 81 + a^2 + 6a + 9 + 4a^2 + 48a + 144 = (a+3)^2 + (2a+12)^2 + (a^2+9a+9)^2$ proving the results for case (I).

The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes.

$\square$

Proposition 2.12. $abcd + 5a^2 + 60a + 261 = \begin{cases} (a+6)^2 + (2a+12)^2 + (a^2+9a+9)^2 \cdots (I), \\ (b+3)^2 + (2b+6)^2 + (b^2+3b-9)^2 \cdots (II), \\ c^2 + (2c)^2 + (c^2-3c-9)^2 \cdots \cdots \cdots (III), \\ (d-3)^2 + (2d-6)^2 + (d^2-9d-63)^2 \cdots (IV). \end{cases}$

has solution in integers if a, b, c, d *are integers such that* $b = a+3, c = a+6, d = a+9$.

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a+3, c = a+6, d = a+9$. Thus, $abcd + 5a^2 + 60a + 261 = a(a+3)(a+6)(a+9) + 5a^2 + 60a + 261 = [a^2(a+6) + 3a(a+6)](a+9) + 5a^2 + 60a + 261 = (a^3 + 6a^2 + 3a^2 + 18a)(a+9) + 5a^2 + 60a + 261 = (a^3 + 9a^2 + 18a)(a+9) + 5a^2 + 60a + 261 = a(a^3 + 9a^2 + 18a) + 9(a^3 + 9a^2 + 18a) + 5a^2 + 60a + 261 = a^4 + 9a^3 + 18a^2 + 9a^3 + 81a^2 + 162a + 5a^2 + 60a + 261 = a^4 + 18a^3 + 99a^2 + 162a + 5a^2 + 60a + 261 = a^4 + 18a^3 + 104a^2 + 212a + 261 \cdots (2.12)$.

Splitting equation (2.12) into sums of three squares we have $a^4 + 18a^3 + 104a^2 + 212a + 261 = a^4 + 18a^3 + 99a^2 + 162a + 81 + a^2 + 12a + 36 + 4a^2 + 48a + 144 = (a+6)^2 + (2a+12)^2 + (a^2+9a+9)^2$ proving the results for cases (I).

The results for case $(II), (III)$ and (IV) follow from case (I) with some slight changes.

$\square$

Proposition 2.13. $abcd + 8a^2 + 4a + 82 = \begin{cases} (2a+1)^2 + (2a)^2 + (9^2+9a+9)^2 \cdots \cdots \cdots (I), \\ (2b+1)^2 + (2b-6)^2 + (b^2+3b-9)^2 \cdots \cdots (II), \\ (2c-11)^2 + (2c-12)^2 + (c^2-3c-9)^2 \cdots (III), \\ (2d-11)^2 + (2d-18)^2 + (d^2-9d-63)^2 \cdots (IV). \end{cases}$

has solution in integers if a, b, c, d *are integers such that* $b = a+3, c = a+6, d = a+9$.

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a+3, c = a+6, d = a+9$. So, $abcd + 8a^2 + 4a + 82 = a(a+3)(a+6)(a+9) + 8a^2 + 4a + 82 = [a^2(a+6) + 3a(a+6)](a+9) + 8a^2 + 4a + 82 = (a^3 + 6a^2 + 3a^2 + 18a)(a+9) + 8a^2 + 4a + 82 = (a^3 + 9a^2 + 18a)(a+9) + 8a^2 + 4a + 82 = a(a^3 + 9a^2 + 18a) + 9(a^3 + 9a^2 + 18a) + 8a^2 + 4a + 82 = a^4 + 9a^3 + 18a^2 + 9a^3 + 81a^2 + 162a + 8a^2 + 4a + 82 = a^4 + 18a^3 + 99a^2 + 162a + 8a^2 + 4a + 82 = a^4 + 18a^3 + 107a^2 + 166a + 91 \cdots (2.13)$.

Breaking equation (2.13) into sums of three squares we have $a^4 + 18a^3 + 107a^2 + 166a + 82 = a^4 + 18a^3 + 99a^2 + 162a + 81 + 4a^2 + 4a + 1 + 4a^2 = (2a+1)^2 + (2a)^2 + (9^2+9a+9)^2$ proving the results for case (I).

The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes. $\square$

Proposition 2.14. $abcd + 13a^2 + 10a + 83 = \begin{cases} (3a+1)^2 + (2a+1)^2 + (a^2+9a+9)^2 \cdots\cdots\cdots (I), \\ (3b-8)^2 + (2b-5)^2 + (b^2+3b-9)^2 \cdots\cdots\cdots (II), \\ (3c-17)^2 + (2c-11)^2 + (c^2-3c-9)^2 \cdots\cdots (III), \\ (3d-26)^2 + (2d-17)^2 + (d^2-9d-63)^2 \cdots\cdots (IV). \end{cases}$

has solution in integers if a, b, c, d *are integers such that* $b = a+3, c = a+6, d = a+9$.

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a+3, c = a+6, d = a+9$. Therefore, $abcd + 13a^2 + 10a + 83 = a(a+3)(a+6)(a+9) + 13a^2 + 10a + 83 = [a^2(a+6) + 3a(a+6)](a+9) + 13a^2 + 10a + 83 = (a^3 + 6a^2 + 3a^2 + 18a)(a+9) + 13a^2 + 10a + 83 = (a^3 + 9a^2 + 18a)(a+9) + 13a^2 + 10a + 83 = a(a^3 + 9a^2 + 18a) + 9(a^3 + 9a^2 + 18a) + 13a^2 + 10a + 83 = a^4 + 9a^3 + 18a^2 + 9a^3 + 81a^2 + 162a + 13a^2 + 10a + 83 = a^4 + 18a^3 + 99a^2 + 162a + 13a^2 + 10a + 83 = a^4 + 18a^3 + 112a^2 + 172a + 83 \cdots (2.14).$
Decomposing equation (2.14) into sums of three squares we have $a^4 + 18a^3 + 112a^2 + 172a + 83 = a^4 + 18a^3 + 99a^2 + 162a + 81 + 9a^2 + 6a + 1 + 4a^2 + 4a + 1 = (3a+1)^2 + (2a+1)^2 + (a^2+9a+9)^2$ proving the results for case (I).
The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes.

□

Proposition 2.15. $abcd + 13a^2 + 6a + 82 = \begin{cases} (3a+1)^2 + (2a)^2 + (a^2+9a+9)^2 \cdots\cdots\cdots\cdots (I), \\ (3b-8)^2 + (2b-6)^2 + (b^2+3b-9)^2 \cdots\cdots\cdots (II), \\ (3c-17)^2 + (2c-12)^2 + (c^2-3c-9)^2 \cdots\cdots (III), \\ (3d-26)^2 + (2d-18)^2 + (d^2-9d-63)^2 \cdots\cdots (IV). \end{cases}$

has solution in integers if a, b, c, d *are integers such that* $b = a+3, c = a+6, d = a+9$.

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a+3, c = a+6, d = a+9$. Therefore, $abcd + 13a^2 + 6a + 82 = a(a+3)(a+6)(a+9) + 13a^2 + 6a + 82 = [a^2(a+6) + 3a(a+6)](a+9) + 13a^2 + 6a + 82 = (a^3 + 6a^2 + 3a^2 + 18a)(a+9) + 13a^2 + 6a + 82 = (a^3 + 9a^2 + 18a)(a+9) + 13a^2 + 6a + 82 = a(a^3 + 9a^2 + 18a) + 9(a^3 + 9a^2 + 18a) + 13a^2 + 6a + 82 = a^4 + 9a^3 + 18a^2 + 9a^3 + 81a^2 + 162a + 13a^2 + 6a + 82 = a^4 + 18a^3 + 99a^2 + 162a + 13a^2 + 6a + 82 = a^4 + 18a^3 + 112a^2 + 168a + 82 \cdots (2.15).$
Splitting equation (2.15) into sums of three squares we have $a^4 + 18a^3 + 112a^2 + 168a + 91 = a^4 + 18a^3 + 99a^2 + 162a + 81 + 9a^2 + 6a + 1 + 4a^2 = (3a+1)^2 + (2a)^2 + (a^2+9a+9)^2$ proving the results for case (I).
The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes.

□

Proposition 2.16. $abcd + 10a^2 + 12a + 91 = \begin{cases} (3a+1)^2 + (a+3)^2 + (a^2+9a+9)^2 \cdots\cdots (I), \\ (3b-8)^2 + b^2 + (b^2+3b-9)^2 \cdots\cdots\cdots\cdots (II), \\ (3c-17)^2 + (c-3)^2 + (c^2-3c-9)^2 \cdots\cdots (III), \\ (3d-26)^2 + (d-6)^2 + (d^2-9d-63)^2 \cdots\cdots (IV). \end{cases}$

has solution in integers if a, b, c, d *are integers such that* $b = a+3, c = a+6, d = a+9$.

Proof. Suppose a, b, c and d are integers whose common difference between any two integer is 3. Then, $b = a+3, c = a+6, d = a+9$. So, $abcd+10a^2+12a+91 = a(a+3)(a+6)(a+9)+10a^2+12a+91 = [a^2(a+6)+3a(a+6)](a+9)+10a^2+12a+91 = (a^3+6a^2+3a^2+18a)(a+9)+10a^2+12a+91 = (a^3+9a^2+18a)(a+9)+10a^2+12a+91 = a(a^3+9a^2+18a)+9(a^3+9a^2+18a)+10a^2+12a+91 = a^4+9a^3+18a^2+9a^3+81a^2+162a+10a^2+12a+91 = a^4+18a^3+99a^2+162a+10a^2+12a+91 = a^4+18a^3+109a^2+174a+91 \cdots (2.16)$.

Decomposing equation (2.16) into sums of three squares we have $a^4+18a^3+109a^2+174a+91 = a^4+18a^3+99a^2+162a+81+9a^2+6a+1+a^2+6a+9 = (3a+1)^2+(a+3)^2+(a^2+9a+9)^2$ proving the results for case (I).

The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes.

$\square$

Proposition 2.17. $abcd + 10a^2 + 18a + 118 = \begin{cases} (3a+1)^2 + (a+6)^2 + (a^2+9a+9)^2 \cdots\cdots (I), \\ (3b-8)^2 + (b-3)^2 + (b^2+3b-9)^2 \cdots\cdots (II), \\ (3c-17)^2 + (c)^2 + (c^2-3c-9)^2 \cdots\cdots\cdots (III), \\ (3d-26)^2 + (d-3)^2 + (d^2-9d-63)^2 \cdots (IV). \end{cases}$

has solution in integers if a, b, c, d *are integers such that* $b = a+3, c = a+6, d = a+9$.

Proof. Suppose a, b, c and d form arithmetic sequence with its common difference 3. Then, $b = a + 3, c = a+6, d = a+9$. Accordingly, $abcd+10a^2+18a+118 = a(a+3)(a+6)(a+9)+10a^2+18a+118 = [a^2(a+6)+3a(a+6)](a+9)+10a^2+18a+118 = (a^3+6a^2+3a^2+18a)(a+9)+10a^2+18a+118 = (a^3+9a^2+18a)(a+9)+10a^2+18a+118 = a(a^3+9a^2+18a)+9(a^3+9a^2+18a)+10a^2+18a+118 = a^4+9a^3+18a^2+9a^3+81a^2+162a+10a^2+18a+118 = a^4+18a^3+99a^2+162a+10a^2+18a+118 = a^4+18a^3+109a^2+180a+118 \cdots (2.17)$.

Breaking equation (2.17) into sums of three squares we have $a^4+18a^3+109a^2+180a+118 = a^4+18a^3+99a^2+162a+81+9a^2+6a+1+a^2+12a+36 = (3a+1)^2+(a+6)^2+(a^2+9a+9)^2$ proving the results for case (I).

The results for cases $(II), (III)$ and (IV) follow from case (I) with some slight changes. $\square$

Conjecture 2.1. $abcd + ka^2 + ma + n = u^2 + v^2 + w^2$ *has no non-negative integer solution if* $b - a = c - b = d - c \neq L$ *for some integer* L.

Conclusion

This Study has introduced and determined partial solution to the sums of three square formula $abcd + ka^2 + ma + n = u^2 + v^2 + w^2$. This was achieved by determining the integer values $a, b, c, d, k, m, n, u, v, w$ for which $abcd + ka^2 + ma + n = u^2 + v^2 + w^2$. The methodology involved use of factorization, modular arithmetics combination with binomial theorem. This far, research in this area is still minimal and we recommend other researchers to do more research on the diophantine equation $abcd + ka^2 + ma + n = u^2 + v^2 + w^2$ in future.

Competing Interests

Authors have declared no competing interest.

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