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Regular Elements and Von-Neumann Inverses of a Class of Zero Symmetric Local Near-Rings Admitting Frobenius Derivations

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Let $\mathcal{N}$ be a zero-symmetric local near-ring. An element $x \in \mathcal{N}$ is either regular, zero or a zero divisor. In this paper, we construct a class of zero symmetric local near-ring of characteristic p^k; $k \geq 3$ admitting an identity frobenius derivation, characterize the structures and orders of the set $R(\mathcal{N})$, the regular compartment with an aim of advancing the classification problem of algebraic structures. The number theoretic notions relating the number of regular elements to Euler's phi-function and the arithmetic functions of Galois near-rings are

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adopted. Using the Fundamental Theorem of finitely generated Abelian groups, the structures of $R(\mathcal{N})$ are proved to be isomorphic to cyclic groups of various orders. The study also extends to the automorphism groups $Aut(R(\mathcal{N}))$ of the regular elements.

Keywords: Regular elements; Von-Neumann inverses; zero symmetric local near-rings.

Subject Classification: 16N60, 16W25, 16Y30.

1 Introduction

The study of near-rings with identity is very vital in generalizing characterization of commutative rings with identity. The original works on near-rings and their applications are attributed to Pilz[1] who have very good foundations upon which these algebraic structures could be advanced. Much of the recent works on the classification of finite rings with identity have however considered a characterization paradigm using the unit groups, the zero divisor graphs, adjacency and incidence matrices among others. This has left the non-linear aspects fairly untouched. In particular, regular elements and Von-Neumann inverses of near rings admitting derivations hardly exist in the available literature.

Oduor, Ojiema and Mmasi[2] determined construction of idealized local rings of characteristic $p^n : \ n = 1, 2, 3$ and determined the structures of the unit groups R^*. Osba, Henriksen and Osama [3] conducted a classification survey on combining local and Von Neumann Regular Rings as a basis upon which the regularity properties of rings and their ideals could be explored. The rings studied in [3] were finite and their Von Neumann inverses gave some asymptotic patterns. Their findings demonstrated how to combine the Von- Neumann inverses of classes of rings such as the power series rings and the ring of integers. They however did not count the number of regular elements in a given finite ring nor did they give the structural formulae for the regular elements and the Von Neumann inverses of the specified classes of rings. In a closely related research, the study on regular elements of Galois rings can be attributed to Osama and Emad [4] where they characterized the regular elements in the ring of integers modulo n, $\mathbb{Z}_n$. Furthermore, they studied the arithmetic functions denoted as $V(n)$ and determined the relationship between $V(n)$ and the Euler's phi function, $\varphi(n)$. This gave an extension of the ring theoretic algebra employed in counting the regular elements of $\mathbb{Z}_n$ to the number theoretic methodologies. For instance, the research revealed that if a is a regular element in $\mathbb{Z}_n$, then $a^{(-1)} \equiv a^{\varphi(n)-1}\ (mod\ n)$. They proposed a criterion for getting the possible Von Neumann inverses in the set of regular elements of $\mathbb{Z}_n$ and explored the asymptotic properties of $V(n)$. Their findings did not consider extensions and idealization using maximal submodules of $\mathbb{Z}_n \forall n \in \mathbb{Z}$.

Closely related works can also be seen in Osba et al [5] and Oduor, Omamo and Musoga[6]. Furthermore, Abujabal et al[7] considered the structure and commutativity of general near-rings. The ideas postulated in [7] were later improved by Asma and Inzamam[8] who gave a number of conditions that determine the commutators and anticommutators of zero symmetric near-rings with Jordan ideals and derivations. Akin[9] studied IFP ideals in near-rings while Ali, Bell and Miyan[10] considered generalized derivations in rings. In order to advance the problem of classification of algebraic structures, the paper discovers new classes of near-rings and classifies them via their regular elements.

2 Zero-Symmetric Local Near-Ring of Characteristic $p^k : \ k \geq 3$

Let $R_o = G\mathcal{N}(p^{kr}, p^k)$. Let $i = 1, ..., h$ and $u_i \in Z_L(\mathcal{N})$ and $\mathcal{M} =< u_i >$.
Then,

$$\mathcal{N} = R_o \oplus \mathcal{M} = R_o \oplus \sum_{i=1}^{h} (R_o/pR_o)^i$$

is a group with respect to addition.

On $\mathcal{N}$, let

$$(r_o, \overline{r}_1, ..., \overline{r}_h)(s_o, \overline{s}_1, ..., \overline{s}_h) = (r_o s_o, r_o \overline{s}_1 + \overline{r}_1 s_o, ..., r_o \overline{s}_h + \overline{r}_h s_o)^{\delta}$$

where δ is the identity Frobenius automorphism. The multiplication turns $\mathcal{N}$ into a local zero symmetric near-ring with identity $(1, \overline{0}, ..., \overline{0})$.

Indeed $\mathcal{N} = R_o \oplus \mathcal{M}$ is commutative since δ is the identity Frobenius automorphism.

Proposition 2.1. *Consider $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k)$ where $k \geq 3$. Then, $char\mathcal{N} = p^k$ and:*

(i). $Z_L(\mathcal{N}) = pR_o \oplus \sum_{i=1}^{h}(R_o/pR_o)^i$

(ii). $(Z_L(\mathcal{N}))^{k-1} = p^{k-1}R_o \neq (0)$

(iii). $(Z_L(\mathcal{N}))^k = (0)$.

Proof. Char $G\mathcal{N}(p^{kr}, p^k) = char\mathcal{N}$ and $id_{\mathcal{N}} = id_{G\mathcal{N}(p^{kr}, p^k)}$
Let $a \in R_o$ and a not contained in pR_0 and let $s \in Z_L(\mathcal{N})$.
Then

$$\begin{aligned} (a+s)^{pr} &= a^{pr} + s' : (s' \in Z_L(\mathcal{N})) \\ &= (a+s'')^{p^r-1} : (s'' \in Z_L(\mathcal{N})) \end{aligned}$$

But $(a+s'')^{p^r-1} \equiv 1 + s'''$ with $s''' \in Z_L(\mathcal{N})$ and $(1+s''')^{p^k-1} = 1$. Hence $(a+s)$ is regular and not zero.
Since $\mid Z_L(\mathcal{N}) \mid = p^{(h+k-1)r}$ and
$\mid (R_o/pR_o)^* + Z_L(\mathcal{N}) \mid = (p^r - 1)(p^{(h+k-1)r})$, it follows that
$(R_o/pR_o)^* + Z_L(\mathcal{N}) = \mathcal{N} - Z_L(\mathcal{N})$ and hence all the elements outside $Z_L(\mathcal{N}) \setminus \{0\}$ are regular. □

Remark 2.1. *A regular element $x \in R(\mathcal{N})$ may have more than one Von-Neumann inverse. However, for the classes of near-rings considered in this study, the Von-Neumann inverses are unique.*

Proposition 2.2. *Let $\mathcal{N}$ be a class of near-ring of the construction. For $x \in \mathcal{N}$ and $x_0 \in I(x)$, where $I(x)$ is the inner inverse set, then:*

$$I(x) = \{x_0 + \alpha - x_0 x \alpha x x_0 \mid \alpha \in \mathcal{N}\}$$

Proof. From the construction, if $x \in \mathcal{N}$, then

$$x = (r_0 + (\sum_{i=1}^{h} r_0 + pr')r' \in G\mathcal{N}(p^{kr}, p^k)/pG\mathcal{N}(p^{kr}, p^k)).$$

So the definition of the multiplication in $\mathcal{N}$ gives the desired result. □

Denote by $l(x)$ and $r(x)$ the left and the right annihilator of an element $x \in \mathcal{N}$. So the inner annihilator of $x \in \mathcal{N}$ is: $Iann(x) = \{y \in N : xyx = 0\}$.

Theorem 2.1. *Let $\mathcal{N}$ be the near ring of the construction. If $a \in R(\mathcal{N})$, then for any $b \in \mathcal{N}$, $bI(a)b$ is a singleton set if and only if $b \in \mathcal{N}a \cap a\mathcal{N}$.*

Proof. Suppose there exists $x, y \in \mathcal{N}$ such that $b = xa = ay$ and let $a_o \in I(a)$. We then have that for any $t \in \mathcal{N}$,

$$\begin{aligned} b(a_o + t - a_o a t a a_o)b &= (x a a_o + x a t - x a t a a_o) a y \\ &= xay + xatay - xatay \\ &= xay \end{aligned}$$

Thus the set $bI(a)b = \{xay\}$ is singleton.

Conversely, suppose that $bI(a)b = \{ba_ob\}$.

We then have: $b(a_o + t - a_oataa_o)b = ba_ob$ for any $t \in \mathcal{N}$. This implies that for any $t \in \mathcal{N}$, we have: $b(t - a_oataa_o)b = 0$..............(i). Substituting $(1 - a_oa)t$ for t in this equality yields $b(1 - a_oataa_o)tb = 0$ for any $t \in \mathcal{N}$. But $\mathcal{N}$ constructed is semiprime so that $b(1 - a_oa) = 0 \Rightarrow b = ba_oa \in \mathcal{N}a$ .....(ii)
Similarly, substituting t by $t(1 - aa_o)$ in the equality (i)
gives $b = aa_ob \in a\mathcal{N}$.........(iii)
Comparing (ii) and (iii), we conclude that $b \in \mathcal{N}a \cap a\mathcal{N}$ □

Lemma 2.1. *Let $\mathcal{N}$ be the near ring constructed and let $b, d \in \mathcal{N}$ such that $b + d$ is a Von Neumann regular element. Then the following are equivalent:*

(i) $b\mathcal{N} \oplus d\mathcal{N} = (b + d)\mathcal{N}$

(ii) $\mathcal{N}b \oplus \mathcal{N}d = \mathcal{N}(b + d)$

(iii) $b\mathcal{N}b \cap d\mathcal{N} = \{0\}$ *and* $\mathcal{N}b \cap \mathcal{N}d = \{0\}$.

The next result shows when $I(a) \subseteq I(b)$ necessarily and sufficiently where $a, b \in \mathcal{N}$

Proposition 2.3. *Let $a, b \in R(\mathcal{N})$. Then $I(a) \subseteq I(b)$ if and only if $b\mathcal{N} \cap d\mathcal{N} = \{0\}$ and $\mathcal{N}b \cap \mathcal{N}d = \{0\}$ where $a = d + b$*

Proof. Let $I(a) \subseteq I(b)$. Then by definition, there exists some $x \in I(a)$ such that $bxb = b$.

Now $b \in \mathcal{N}a \cap a\mathcal{N}$.

Write $b = \alpha a = a\beta$ where $\alpha, \beta \in \mathcal{N}$.

Then $bI(a)a = b$.

Next

$$\begin{aligned} bI(a)d &= bI(a)a - bI(a)b \\ &= b - bI(a)b = 0 \end{aligned}$$

Consider now

$$\begin{aligned} dI(a)b &= aI(a)b - bI(a)b \\ &= \alpha\beta - bI(a)b \\ &= b - b = 0 \end{aligned}$$

We thus have $bI(a)d = 0$ and $dI(a)b = 0$...(i)

Then for any $x \in I(a)$ we have;

$$\begin{aligned} b + d = a &= axa \\ &= (b + d)x(b + d) \\ &= bxa + dxb + dxd \\ &= b + 0 + dxd \end{aligned}$$

This yields $dI(a)d = d$...(ii)

To show that $d\mathcal{N} \cap b\mathcal{N} = \{0\}$.

Let $bx = dy \in b\mathcal{N} \cap d\mathcal{N}$.

Multiplying both sides of (ii) by y on the right and using $bx = dy$ yields, $dI(a)bx = dy$

But from above we have that $dI(a)b = 0$ and so $dy = 0$ which clears the proof.

Similarly, we show that $\mathcal{N}b \cap \mathcal{N}d = \{0\}$.

Let $xb = yd \in \mathcal{N}b \cap \mathcal{N}d$. Multiplying both sides of (ii) on the left by y. We get:

$ydI(a)d = yd$. This proves that $xbI(a)d = yd$.

Since $bI(a)d = 0$, we obtain $yd = 0$ showing that $\mathcal{N}b \cap \mathcal{N}d = \{0\}$.

□

Theorem 2.2. *Let $a, b \in R(\mathcal{N})$. Then $I(a) = I(b)$ if and only if $a = b$.*

Proof. From the construction, $\mathcal{N} = Z_L(\mathcal{N}) \cup \mathcal{N}^* \cup \{0\}$. Now, assume that $I(a) = I(b)$, we can write $a = b + d$ with $b\mathcal{N} \cap d\mathcal{N} = 0$ and $\mathcal{N}d \cap \mathcal{N}d = 0$. But $(b + d)\mathcal{N} = b\mathcal{N} \oplus d\mathcal{N}$. Since $I(a) = I(b)$, we have that $aI(b)a = \{a\}$ and $bI(a)b = \{b\}$ and therefore it follows that $\mathcal{N}a = \mathcal{N}b$ and $a\mathcal{N} = b\mathcal{N}$ which leads to $a\mathcal{N} = (b+d)\mathcal{N} = b\mathcal{N} \oplus d\mathcal{N}$ giving $d = 0$. Hence $a = b$ as desired.
□

Next, we provide the analogue to the previous theorem by generalizing the case to reflexive inverses:

Theorem 2.3. *Let $a, b \in R(\mathcal{N})$. Then $Ref(a) = Ref(b)$ iff $a = b$*

Proof. Let $a_o \in Ref(a) = Ref(b)$. Since $a = 0$ if and only if $Ref(a) = 0$, assume that $a, b \neq 0$.Since $bRef(a)b = bRe(b)b = b$ and $Ref(a) = I(a)aI(a)$, we have that for any $t \in N$. $b(a_o + t - a_oataa_o)a(a_o + t - a_oataa_o)b = b$. Replacing t by $(1 - a_oa)t$ and noting that $a(1 - a_oa) = 0$,we obtain successively
$b(a_oa + (1 - a_oa)ta)(a_o + (1 - a_oa)t)b = b$ and $b(a_ob + (1 - a_oa)ta)(a_o)b = b$ and so $ba_ob + b(1 - a_oa)taa_o)b = b$. Since $ba_ob = b$ gives $b(1 - a_oa)taa_ob = 0 \ \forall \ t \in \mathcal{N}$, this leads to $aa_ob(1 - a_oa)taa_ob(1 - a_oa) = 0 \ \forall \ t \in \mathcal{N}$.
But we are guaranteed of semi-primeness od $\mathcal{N}$ which then implies that $aa_ob(1 - a_oa) = 0$. Left multplying by $a_o \in Ref(a)$, we get that
$a_ob(1 - a_oa) = 0$ and hence since $a_o \in I(b)$, we conclude that $b(1 - a_oa) = 0$.
Therefore we obtain that $\mathcal{N}b \subseteq \mathcal{N}a$ and $\mathcal{N}a \subseteq \mathcal{N}b$ which implies that $\mathcal{N}a = \mathcal{N}b$.
□

3 Structures and Orders of Von-Neumann Regular Elements

Definition 3.1. *Let $(\mathcal{N}, +)$ be a group. The exponent of the group is the least common multiple of all the orders of the group elements.*

Remark 3.1. *Let N be a finite near-ring with identity 1 and n be the exponent of $(\mathcal{N}, +)$. Then $ord(1) = n$.*

Let $\mathbb{Z}_n$ be the ring of integers modulo n. Then $| \mathbb{Z}_n^* | = \varphi(n)$, φ- being the Euler-Phi function. We now give a generalization of this result to an arbitrary case:

Proposition 3.1. *Let $\mathcal{N}$ be the near-ring from classes of near-rings in construction I and II and $\mathcal{N}^*$ be as obtained in the constructions. Let n be the exponent of $(\mathcal{N}, +)$ and φ be the Euler's-Phi function. Then there is a subgroup of order $\varphi(n)$ contained in $\mathcal{N}^*$.*

Proof. We use the fact that the identity $(1, 0, 0, ..., 0) \in \mathcal{N}$ generates a subring of $\mathcal{N}$. Assume the usual $(+)$ and the multiplication $(.)$ defined on $\mathcal{N}$. Consider the cyclic group $< 1, 0, 0, ..., 0 >$, additively generated by 1 where $1 \equiv (1, 0, 0, ..., 0)$. Then $l.1 = \underbrace{1 + 1 + ... + 1}_{l}$ and $k.1 = \underbrace{1 + 1 + ... + 1}_{k}$ are two elements of $< 1 >$. Since 1 is an identity: $(l.1)(k.1) = (lk.1) \in < 1 >$. Thus $S = (< 1 >, +, .)$ is a sub-near ring containing the identity. Indeed $f : S \longrightarrow \mathbb{Z}_n : f(k.1) = [k]_n$ is a near-ring isomorphism. Thus $\cong \mathbb{Z}_n$. Let S^* be the group of units of S. It follows from the canonical isomorphism above that S^* has $\varphi(n)$ invertible elements. Since S and N have the same identity elements, an element $y \in S : y^{-1} \in S$ implies that $y^{-1} \in N$
$\therefore S^* \subseteq N^*$ and S^* is a subgroup of order $\varphi(n)$. ☐

In the sequel, we recall some notions in Number Theory: Let $\mathcal{N} = \mathbb{Z}_{p^k}$. For each natural number n, we have the following functions:

$\varphi(n) = \{\sharp x : 1 \leq x \leq n \ \gcd(x, n) = 1\}$, $\overline{w}(n)$ = number of distinct primes dividing n, $\tau(n)$ = number of the divisors of n and $\sigma(n)$ = sum of the divisors of n.
For example if $p = 2$ and $k = 2 \Rightarrow n = 4$, then: $\varphi(4) = 2$, $\overline{w}(4) = 1$, $\tau(4) = 3$ and $\sigma(4) = 1 + 2 + 4 = 7$

Theorem 3.1. *([4], Theorem 2) Let p be a prime integer and $k \in \mathbb{Z}^+$ then $a \in G\mathcal{N}(p^k, p^k)$ is regular if $a^{p^k - p^{k-1}+1} \equiv a(mod\, p^k)$*
The element $a^{p^k - p^{k-1}+1}$ is a Von Neumann inverse of a

Example 3.1. *Let $\mathcal{N} = \mathbb{Z}_4[x]/< x + 1 >$. Then $\mathcal{N} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. By definition, an element a is a memeber of $R(\mathcal{N})$ if and only if $a^{p^k - p^{k-1}+1} \equiv a(mod\, p^k)$. Thus, if $a = \bar{3}$, then, $\bar{3}^{2^2 - 2^{2-1}+1} \equiv \bar{3}(mod\ 4)$ which implies that $(\bar{3})^3 \equiv \bar{3}(mod\ 4)$*
Therefore, $\bar{3}$ is a regular element and $(\bar{3})^3$ is a Von-Neumann inverse. So, the Von-Neumann inverses of $\bar{1}$, $\bar{3}$ are $\bar{1}$, $\bar{3}$ respectively

Theorem 3.2. *Let $\mathcal{N} = G\mathcal{N}(p^k, p^k)$. Then,*

$$V(p^k) = p^k - p^{k-1} + 1 = \varphi(p^k) + 1.$$

Proof. Since $\mathcal{N} = G\mathcal{N}(p^k, p^k)$ is zero-symmetric local, every element $a \in R(\mathcal{N})$ is either 0 or a unit.
But $| \mathcal{N}^* : p^{k-1} + 1$ and the zero element is unique, it follows from the arithmetic function formula that:

$$V(p^k) = p^k - p^{k-1} + 1 = \varphi(p^k) + 1.$$

☐

Definition 3.2. *Let $x, y \in \mathbb{Z}^+$. We say that x is a unitary divisor of y if $x \mid y$ and $\gcd(x, \frac{y}{x}) = 1$ and we write $x \| y$.*

The number of regular elements in $\mathcal{N}$ can then be calculated using the unitary divisors of an integer $n =| \mathcal{N} |$

Proposition 3.2. *Let $\mathcal{N} = G\mathcal{N}(p^k, p^k)$. Then $V(\mathcal{N}) = \Sigma_{x \| p^k} \varphi(x)$ and $V(N)/\varphi(p^k) = \Sigma_{x \| p^k} \frac{1}{\varphi(x)}$*

Proof. In $\mathcal{N}$ above $x = 1$ and $x = p^k \equiv 0(mod p^k)$.
By definition, $\varphi(1) = 1$. But $\varphi(p^k) = p^k - p^{k-1}$ and

$$\begin{aligned} V(p^k) &= p^k - p^{k-1} + 1 \\ &= \varphi(p^k) + \varphi(1) \end{aligned}$$

Moreover,

$$\begin{aligned}\frac{V(p^k)}{\varphi(p^k)} &= \frac{p^k - p^{k-1} + 1}{p^k - p^{k-1}} \\ &= 1 + \frac{1}{p^k - p^{k-1}} \\ &= \frac{1}{\varphi(1)} + \frac{1}{\varphi(p^k)}\end{aligned}$$

The summatory function:

$$\begin{aligned}K(p^k) &= \sum_{x\|(p^k)} V(x) \\ &= \sum_{i=0}^{k} V(p^i) \\ &= V(1) + \sum_{i=1}^{k} V(p^i) \\ &= V(1) + \sum_{i=1}^{k} [(p^i - p^{i-1}) + 1] \\ &= 1 + (p + p^2 + ... + p^k) - (1 + p + p^2 + ... + p^{k-1}) + k\end{aligned}$$

$K(p^k) = p^k + k$ □

Example 3.2. *Consider $\mathcal{N} = GR(2^2, 2^2)$, then*

$$\begin{aligned}V(2^2) &= \sum_{t||} \varphi(t) \\ &= \varphi(1) + \varphi(4) \\ &= 1 + 2 = 3.\end{aligned}$$

Thus the number of regular elements are 3.

Theorem 3.3. *Let $\mathcal{N} = GR(p^k, p^k)$ and $\sigma(p^k)$ be the sums of the divisors of p^k. Then*

$$\begin{aligned}\sigma(p^k) &= \sum_{i=0}^{k} p^i and \\ V(p^k)\sigma(p^k) &= [p^k - p^{k-1}][\sum_{i=0}^{k} p^i]\end{aligned}$$

Proof. Clearly,

$$
\begin{aligned}
V(p^k)\sigma(p^k) &= [p^k - p^{k-1}][\sum_{i=0}^{k} p^i] \\
&= p^k(1 - \frac{1}{p} + \frac{1}{p^k})(\sum_{i=1}^{k} p^i) \\
&= p^k(1 - \frac{1}{p} + \frac{1}{p^k})(1 + p + p^2 + ... + p^k) \\
&= p^k[1 + p + p^2 + ... + p^k - \frac{1}{p} - 1 - p - ...p^{k-1} + \frac{1}{p^k} + \frac{1}{p^{k-1}} + \frac{1}{p^2} + \frac{1}{p} + 1] \\
&= p^k[1 + p^k + p^{-2} + p^{-3} + ... + p^{2-k} + p^{1-k} + p^k] \\
&= p^k[1 + p^k + \sum_{i=2}^{k} p^{-i}] \\
&= p^{2k}[1 + p^{-k} + \sum_{i=2}^{k} p^{-(k+i)}]
\end{aligned}
$$

which implies that

$$
\frac{V(p^k)\sigma(p^k)}{p^{2k}} = 1 + p^{-k} + \sum_{i=2}^{k} p^{-(k+i)}
$$

as required □

Theorem 3.4. *Let $\mathcal{N} = GR(p^k, p^k)$. Then $\sigma(p^k) + \varphi(p^k) \leq p^k\tau(p^k)$*

Proof. Let $k = 1$. Then $\sigma(p^k) = p + 1$ and $\varphi(p) = p - 1$ so that $\sigma(p) + \varphi(p) = 2p$. Since p has only two divisors 1 and p, this implies that $2p = p(p\tau)$. Thus $\sigma(p) + \varphi(p) = 2p$. Now suppose that $k > 1$, then,

$$
\sigma(p^k) = \sum_{i=1}^{k} p^i
$$

and $\varphi(p^k) = p^k - p^{k-1}$ so that

$$
\begin{aligned}
\sigma(p^k) + \varphi(p^k) &= 1 + p + ... + p^k + p^k + p^{k-1} \\
&= 2p^k + p^{k-2} + ... + p + 1 < (k+1)p^k
\end{aligned}
$$

But p^k has $(k+1)$ divisors so that $(k+1)p^k = p^k\tau(p^k)$
thus $\sigma(p^k) + \varphi(p^k) < p^k\tau(p^k)$ □

Example 3.3. *Let $\mathcal{N} = \mathbb{Z}_4[x]/ < x + 1 >= GR(2^2, 2^2)$*

$$
\begin{aligned}
\sigma(2^2) + \varphi(2^2) &\leq 2^2\tau(2^2) \\
\Rightarrow \sigma(4) + \varphi(4) &\leq 4\tau 4 \\
\Rightarrow 7 + 2 &\leq 4 \times 3.
\end{aligned}
$$

Thus the result of $\sigma(p^k) + \varphi(p^k) < p^k\tau(p^k)$ holds.

Proposition 3.3. *Consider $\mathcal{N} = GR(p^{kr}, p^k)$ where $kr = n > 1$. Then $\sigma(p^n) + V(p^n) < p^n\tau(p^n)$*

Proof. $1 + \frac{1}{p} + \frac{1}{p^2} + ... + p^n < n = (n+1) - 1 = \tau(p^n) - 1$ Now

$$\begin{aligned} \frac{\sigma(p^n)}{p^n} &= \frac{1 + p + p^2 + ... + p^n}{p^n} < \tau(p^n) - 1 \\ \Rightarrow \sigma(p^n) &< \sigma p^n[\tau(p^n) - 1] \\ &= p^n\tau(p^n) - p^n \end{aligned}$$

Since $V(p^n) < p^n$, we clear that $\sigma(p^n) + V(p^n) < p^n\tau(p^n)$. However, if $n = 1$, then $\sigma(p) + V(p) > p\tau(p)$. Let

$$\begin{aligned} \mathcal{N} &= \mathbb{Z}_2[x]/ < x^2 + x + 1 >: p = 2, r = 2, k = 1, n = kr > 1 \\ &= \{\overline{0}, \overline{1}, \overline{x}, \overline{x+1}\} \end{aligned}$$

We notice that,

$$\begin{aligned} \sigma(p) &= \sigma(2) = 1 + 2 = 3 \\ V(p) &= V(2) = 2 \\ \tau(p) &= \tau(2) = 2 \\ \Rightarrow \sigma(p) + V(p) > p\tau(p) i.e. 5 > 4. \end{aligned}$$

But, if $\mathcal{N} = \mathbb{Z}_2[x]/ < x^2 + x + 1 > \cong GR(p^{kr}, p^k), k = 2, r = 2, p = 2,$
$\sigma(p^k) = \sigma(4) = 2$, $V(4) = 4$, $p^k\tau(p^k) = 4\tau(4) = 4 \times 3 = 12$

Therefore $\sigma(p^k) + V(p^k) < p^k\tau(p^k)(6 < 12)$ which justifies the previous result. □

Lemma 3.1. *Let $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k) \oplus \mathcal{M}$ where p is prime k and r are positive integers and $\mathcal{M}$ is a h-dimensional module over $\mathcal{N}$. Then if $h = 0$,*

(i) $R(\mathcal{N}) \cong (1 + Z(\mathcal{N})) \cup \{0\}$ *and*

(ii) $| R(\mathcal{N}) | = (p^{(k-1)r})(p^r - 1) + 1$

Proof. Let $a \in R(\mathcal{N}) \cong (1 + Z(\mathcal{N}))$. Then a is invertible or 0. But $\mathcal{N}$ is local means that a is regular i.e. $a \in R(\mathcal{N})$.

Thus $R(\mathcal{N}) \subseteq [< a > \times 1 + Z(\mathcal{N}))] \cup \{0\}$..(i)

Conversely, let $a \in R(\mathcal{N})$. Then by definition $\exists$ an element $b \in R(\mathcal{N})$ such that $a = a^2b \Rightarrow a(1 - ab) = 0$.

If $a \in (\mathcal{N}^*)$ then $1 - ab = 0 \Rightarrow ab = 1$.

Hence b is a Von Neumann inverse of a. If is not a member of $\mathcal{N}^*$ then ab is not a member of $\mathcal{N}^*$ but $ab = aabb = a^2b^2 = abab = (ab)^2$.

Since $\mathcal{N}$ commutes $\Rightarrow ab = (ab)^2 \Rightarrow ab(1 - ab) = 0$.

Now $\Rightarrow 1 - ab$ is a unit and $ab = 0$ so that $a = 0$ because b is its Von Neumann inverse.

$[\{< a > \times 1 + Z(\mathcal{N})\} \cup \{0\}] \subseteq R(\mathcal{N})$..(ii)

Combining (i) and (ii) gives

$$\begin{aligned} R(\mathcal{N}) &\cong [1 + Z(\mathcal{N})] \cup \{0\} \\ &= < a > \times [1 + Z(\mathcal{N})] \cup \{0\} \end{aligned}$$

Next,

$$\begin{aligned}\mathcal{N}^* &= (\mathcal{N}^*/1+Z(\mathcal{N}))\times 1+Z(\mathcal{N})\\ &\cong <a> \times[1+Z(\mathcal{N})]\\ &= \mathbb{Z}_{p^r-1}\times[1+Z(\mathcal{N})]\end{aligned}$$

But

$$\begin{aligned}\mid [1+Z(\mathcal{N})]\mid &= \mid Z(\mathcal{N})\mid\\ &= p^{(k-1)r}\end{aligned}$$

Therefore $\mid \mathcal{N}^* \mid = (p^r-1)(p^{(k-1)r})$
But $R(\mathcal{N}) = \mathcal{N}^* \cup \{0\}$ $\mid R(\mathcal{N}) \mid = (p^r-1)(p^{(k-1)r})+1$ as required. □

Theorem 3.5. *Let $\mathcal{N}$ be the near-ring constructed and $R(\mathcal{N})$ be the set of all the regular elements. Then*

$$R(\mathcal{N}) = \begin{cases} \mathbb{Z}_{2^r-1}\times\mathbb{Z}_2\times\mathbb{Z}_{2^{k-2}}\times\mathbb{Z}^{r-1}_{2^{k-1}}\times(\mathbb{Z}_2)^h\cup\{0\} & p=2;\\ \mathbb{Z}_{p^r-1}\times\mathbb{Z}^r_{p^k-1}\times(\mathbb{Z}^r_p)^h\cup\{0\} & p\neq 2: Char\mathcal{N}=p^k : k\geq 3.\end{cases}$$

Proof. Let char $\mathcal{N} = p^k : k \geq 3$. We provide the general case using $p =$ odd.

Notice that every $l = 1, ..., r; (1 + p\tau_1)^{p^{k-1}} = 1$

$$(1+\tau_l u_1)^{p^k} = 1, ..., (1+p\tau_L u_1+\tau_l u_2+...+\tau_l u_n)^{p^k} = 1.$$

Let $a_l, b_{1l}, ..., b_{hl} \in \mathbb{Z}^+$ with $a_l \leq p^{k-1}$, $b_{il} \leq p^k : 1 \leq i \leq h$. We notice that

$$\prod_{l=1}^{r}\{(1+p\tau_L)^{a_L}\}\cdot\prod_{l=1}^{r}\{(1+\tau_l u_1)^{b_{1l}}\}\cdot\prod_{l=1}^{r}\{(1+\tau_l u_1+\tau_l u_2+...+\tau_l u_h)\} = 1$$

which implies that $a_l = p^{k-1}$, $b_{1l} = p^k = \cdots = b_{hl} = p^k$. Set

$$\begin{aligned}T_l &= <\{(1+p\tau_l)^a \mid a=1,...,p^{k-1}\}>\\ S_{1l} &= <\{(1+\tau_l u_1)^{b_1} \mid b_1=1,\cdots,p^k\}>\\ &\vdots\\ S_{hl} &= <\{(1+\tau_l u_1+\cdots+\tau_l u_n)^{b_h} \mid b_h=1,\cdots,p^k\}>\end{aligned}$$

The sets defined are all cyclic subgroups of the group $1+Z(\mathcal{N})$ and they are of the indicated orders. Furthermore, the intersection of any pair of the cyclic subgroups indicated gives an identity group and the product of the $(h+1)r$ subgroups gives:

$\mid T_l \times S_{1L} \times S_{hl} \mid = p^{k((h+1)r)-1}$ exhausting $1 + Z(\mathcal{N})$.

Thus $1 + Z(\mathcal{N}) \cong \mathbb{Z}^r_{p^{k-1}} \times (\mathbb{Z}^r_p)^h$.

Therefore

$$\begin{aligned}R(\mathcal{N}) &=< \alpha > \ltimes(1+(Z(\mathcal{N})))\cup\{0\}\\ &= \mathbb{Z}_{p^r-1}\times\mathbb{Z}^r_{p^k-1}\times(\mathbb{Z}^r_p)^h\cup\{0\}.\end{aligned}$$

□

Theorem 3.6. *Let $\mathcal{N} = R_o \oplus \mathcal{M}$ where $r = 1$ and p-prime, $k \in \mathbb{Z}^+$. If $\mathcal{M} = R_0/pR_0 \oplus ... \oplus R_0/pR_0$. Let $r_0 \in R(R_0)$ then, its Von-Neumann inverse is*
$r_0^{-1} = r_0^{p^k - p^{k-1} - 1}$ *and* $(r_0, ..., r_h)^{-1} = (r^{p^k - p^{k-1} - 1}, -r_1 t_0 r_0^{-1}, ..., -r_h t_0 r_0^{-1})$

Proof. We know that if $a \in R_0 = G\mathcal{N}(p^{kr}, p^k)$ and $a \in R_0$ then, the Von-Neumann inverse of a is given by: $a^{-1} \equiv a^{p^{(k-1)r}(p^r - 1)}(mod p^k)$ therefore

$$r_0^{-1} \equiv r_0^{p^k - p^{k-1} - 1}$$

as required in step 1

Now let $(t_0, ..., t_h) = (r_0, ..., r_h)^{-1}$, then

$$\begin{aligned} (r_0, r_1, ..., r_h) &= (r_0, ..., r_h)^2 (t_0, ..., t_h) \\ &= (r_0^2, r_0 r_1 + r_1 r_0, ..., r_0 r_h + r_h r_0)(t_0, ..., t_h) \\ &= (r_0^2 t_0, r_0^2 t_1 + (r_0 r_1 + r_1 r_0) t_0, ..., r_0^2 t_h + (r_0 r_h + r_h r_0) t_0) \end{aligned}$$

therefore $r_0 = r_0^2 t_0 \Rightarrow r_0 t_0 = 1 \Rightarrow t_0 = r_0^{-1} = r_0^{p^k - p^{k-1} - 1}$

For $i = 1, ..., h, r_i = r_0^2 t_i + (r_0 r_i + r_i r_0) t_0$

$$\begin{aligned} \Rightarrow r_0^2 t_i &= r_i - (r_0 r_i + r_i r_0) t_0 \\ \Rightarrow \quad t_i &= \frac{r_i - 2 r_0 r_i t_0}{r_0^2} (\therefore \mathcal{N} commutative) \\ \Rightarrow t_i &= \frac{r_i}{r_0^2} - \frac{2 r_i t_0}{r_0} \end{aligned}$$

But $t_0 = r_0^{-1}$

$$\begin{aligned} \Rightarrow t_i &= \frac{r_i}{r_0^2} - \frac{2 r_i}{r_0^2} \\ &= -\frac{r_i}{r_0^2} = -r_i r_0^{-2} \end{aligned}$$

$\therefore t_1 = -r_1 r_0^{-2} ... t_h = -r_h r_0^{-2}$
$\Rightarrow (r_0, ..., r_h)^{-1} = (r_0^{p^k - p^{k-1} - 1}, ..., -r_h r_0^{-2})$ as required □

Example 3.4. $\mathcal{N} = \mathbb{Z}_9 \oplus \mathbb{Z}_9/3\mathbb{Z}_9 \oplus ... \oplus \mathbb{Z}_9/3\mathbb{Z}_9$
Then

$$\begin{aligned} (2, \overline{2}, ..., \overline{2})^{-1} &= (2^{9-3-1}, (-2)(5)^2, ..., (-2)(5)^2) \\ &= (5, \overline{1}, \overline{1}, ..., \overline{1}) \end{aligned}$$

$(5, \overline{1}, \overline{1}, ..., \overline{1})(2, \overline{2}, ..., \overline{2}) = (1, \overline{0}, ..., \overline{0})$

Example 3.5. *Consider* $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k) \cong \mathbb{Z}_2[x] / < x^2 + x + 1 >$ *where* $p = 2, k = 1, r = 2$.

Now $G\mathcal{N} = \{0, 1, x, x+1\}$ *and* $R(\mathcal{N}) = \{0, 1, x, x+1\}$.

Let $\mathcal{N} = G\mathcal{N}(4, 2) \oplus G\mathcal{N}(4, 2)$ *with* $G\mathcal{N}(4, 2)$ *as defined above, then:*

$$\mathcal{N} = \{0, 1, x, x+1\} \oplus \{0, 1, x, x+1\}$$

$= \{(0,0),(0,1),(0,x),(0,x+1),(1,0),(1,1),(1,x),(1,x+1),(x,0),(x,1),(x,x),$
$(x,x+1),(x+1,0),(x+1,1),(x+1,x),(x+1,x+1)\}$
So $|\mathcal{N}| = 16$, $Z_L(\mathcal{N}) = \{(0,0),(0,1),(0,x),(0,x+1)\}$. *Since* $\mathcal{N}$ *is an extension of* $G\mathcal{N}(4,2)$,

$$|R(N)| = 13 = (p^r - 1)(p^{kr}) + 1$$

Applying $(r_0, r_1)^{-1} = (r_0^{p^k - p^{k-1} - 1}, -r_1 r_0^{-2})$, *we can find the Von Neumann inverses of all the members of* $R(\mathcal{N})$.

For instance,

$$R(\mathcal{N}) = \{(1,0),(1,1),(1,x),(1,x+1),(x,0),(x,1),(x,x),(x,x+1),$$
$$(x+1,0),(x+1,1),(x+1,x),(x+1,x+1)\}.$$

So $(1,0)^{-1} = (1^{2^1 - 2^0 - 1}, -01^{-1}) = (1^2, 0) = (1,0)$, $(x,x)^{-1} = (x^{-2}, x^{-1})$

This can be done in the same manner for the other members of $R(\mathcal{N})$. *The next result gives the structures and orders of the automorphism groups of the regular elements,* $R(\mathcal{N})$.

Theorem 3.7. *Let* $\mathcal{N}$ *be a near-ring of construction* $R(\mathcal{N})$ *be the set of all the regular elements including 0. Then if*
$Aut : R(\mathcal{N}) \to R(\mathcal{N})$ *we have that*

$$Aut(R(\mathcal{N})) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{(k-1)r}(GN(p^{kr}, p^k))] \times GL_{hr}(GN(p^{kr}, p^k))] \cup \triangle$$

Theorem 3.8. *Let* $\mathcal{N}$ *be a zero symmetric local near-rings from the class of near-rings of the construction. Then:*

$$|Aut(R(\mathcal{N}))| = [\varphi(p^r - 1) \cdot \prod_{k=1}^{(k-1)r} (p^k - p^{k-1}) \cdot \prod_{k=1}^{hr} (p^k - p^{k-1})] + 1$$

when $char\mathcal{N} = p^k : k \geq 3$

4 Conclusion

This study was set up with an aim of determining and classifying the regular elements and Von-Neumann inverses of the zero symmetric local near-rings with n-nilpotent radical of Jordan ideals admitting Frobenius derivations. The study gave a general construction representing the classes of the near-rings under investigations whose algebraic structures assumed commutation checks attributed the Theorems of Asma and Inzamam in [8] . The structures and orders of $R(\mathcal{N})$ were then characterized in a case by case basis using the Fundamental Theorem of Finitely Generated Abelian Groups and the properties of the general linear groups in the endomorphism of $R(\mathcal{N})$ respectively. The structures of $V(|R(\mathcal{N})|)$ followed asymptotic patterns proposed by Osama and Emad [4] using the properties of $V(n)$, $\tau(n)$, $\overline{\omega}(n)$, $\sigma(n)$ and $K(n)$. The results reveal unique algebraic structures.

Competing Interests

Authors have declared that no competing interests exist.

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