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EFFECT OF A FORCING TERM ON SOLUTION OF A HOMOGENEOUS BURGERS EQUATION

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Abstract—

The importance of the Burgers' equation in engineering can never be overlooked. This equation plays a very crucial role when it comes to models in mathematics and it's applicable in solution of mathematical problems both in applied mathematics and engineering at large. It models mathematical problems that are of a fluid flow nature. In this paper we study the effect of a forcing term on solutions of a homogeneous Burgers' equation using Galerkin mixed finite element method with Robin's boundary conditions. It's observed that the forcing term produces solutions that are both convergent and divergent but most importantly is the fact that these solutions observe some kind of steady behaviour.

Mathematics Subject Classification: Primary 65N30;

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***Keywords*—:** **Burgers' equation, Forcing term, Diffusive term, Convective term, Robin's boundary conditions, Finite element,**

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I. INTRODUCTION

Burgers' equation was proposed as a mathematical model of turbulence [3]. Since then, this model equation has been found applicable in many disciplines such as number theory, gas dynamics, heat conduction, elasticity etc [8]. Since Burgers' equation involves non linear advection term and dissipation term, it is used to simulate wave motion. Comparison of numerical solutions can be made due to having analytical solutions of the Burgers' equation. Various numerical techniques have been applied to solve the equation. The Burgers' equation is a non linear partial differential equation of second order as follows

$$\frac{\partial w}{\partial t}(t,x)+w(t,x)\frac{\partial w}{\partial x}(t,x)=\varepsilon\frac{\partial^2 w}{\partial x^2}(t,x)+f(t,x) \quad (1)$$

Where $\frac{\partial w}{\partial t}(t,x)$ is the dependent variable

$w(t,x)\frac{\partial w}{\partial x}(t,x)$ Is the convective term

$\frac{\partial^2 w}{\partial x^2}(t,x)$ Is the diffusive term

$f(t,x)$ is the forcing term

ε Is the viscosity constant.

It is used in fluid dynamics teaching and in engineering as a simplified model for turbulence, boundary layer behaviour, shock wave formation and mass transport. It has been studied and applied for many decades. Many different closed form, series approximations and numerical solutions are known for particular set of boundary conditions. The study of the properties of Burgers' equation has attracted considerable attention due to its application in various theories. The equation plays a very vital role both for the conceptual understanding of a class of physical flows and for testing various numerical algorithms. A great deal of effort has been expended in the past years to compute efficiently the numerical solutions of this great equation. The Burgers' equation has been used to study a number of physically important phenomena, including shock waves, acoustic transmission and traffic flow [3]. Besides its importance in understanding convection-diffusion phenomena, Burgers' equation can be used, especially for computational purposes for fluid flow problems [18].

In spite the fact that the numerical solution of Burgers' equation has received a fair amount of attention it seems to draw more and more attention every day. Actually some of the importance of Burgers' equation stems from the fact that it is one of the few non-linear equations with known exact solutions in low dimensions.

However the equation gets its name from the extensive research of Burgers' [4] beginning in 1939. He basically focused on modeling turbulence, but the equation is useful in modeling such diverse physical phenomena as shock flows, traffic flow, acoustic transmission in fog etc. In fact, it can be used as a model for any non-linear wave propagation problem subject to dissipation.

Depending on what one is modeling this dissipation may result from viscosity, heat conduction, mass diffusion, thermal radiation, chemical reaction, or other source. We intend to take the point of view that Burgers' equation is a perturbation of the linear heat equation.

In particular, we consider the Burgers' Equation without a forcing term

$$\frac{\partial w}{\partial t}(t,x)+w(t,x)\frac{\partial w}{\partial x}(t,x)=\varepsilon\frac{\partial^2 w}{\partial x^2}(t,x) \qquad (2)$$

Where $\varepsilon > 0$ is a viscosity constant in most problems. In this paper, ε is the thermal diffusivity.

There are several attempts that have been made to solve different kinds of Burgers' equation as discussed below.

Burgers' equation was first studied by Bateman [3] and then studied extensively by Burgers' [4] Fletcher also illustrated some of the dynamics and difficulties of solving the Burgers' equation.

Pugh [16] compares the Galerkin finite element method and the Garlerkin/conservation method in solving the homogenous Burgers' equation

$$\frac{\partial w}{\partial t}(t,x)+w(t,x)\frac{\partial w}{\partial x}(t,x)=\varepsilon\frac{\partial^2 w}{\partial x^2}(t,x) \qquad (3)$$

With homogenous Neumann boundary conditions on the interval [0, 1]

$$\frac{\partial w}{\partial x}(t,0)=0\ ;\ \frac{\partial w}{\partial x}(t,1)=0 \qquad (4)$$

With initial condition

$$w(0,x)=w_0(x) \qquad (5)$$

In the conservation method, the homogenous Burgers' equation is written as

$$\frac{\partial w}{\partial t}(t,x)+\frac{1}{2}\frac{\partial}{\partial x}(w^2(t,x))=\varepsilon\frac{\partial^2 w}{\partial x^2}(t,x) \qquad (6)$$

He further discovered that the Galerkin/conservation method computed more quickly with less error than the Galerkin finite element method. In some cases the Galerkin/conservation method converged to a solution when the Galerkin finite element method diverged. Byrnes, Gilliam, and Shubov [6] proved that for a sufficiently small initial condition, the solution of the Burgers' equation with Neumann boundary conditions must converge to a constant as time approaches infinity. Burns et. al. later showed that this was true for any initial condition, provided the steady-state limit exists. Pugh [16], however, obtained non-constant steady-state solutions for larger initial conditions and smaller ε ($R_e = 120, 240$).

Huanzen Chen and Ziwen Jiang [10] in their paper propose a new mixed finite element method called the characteristics-mixed method for approximating the solution to Burgers' equation. The hyperbolic part of the equation is approximated along the characteristics in time and the diffusion part of the equation is approximated by a mixed finite element method of lowest order. The scheme is locally conservative since fluid is transported along the approximate characteristics on the discrete level and the test function can be piece-wise constant. Their analysis showed that the new method approximate the scalar unknown and the vector flux optimally and simultaneously. They also showed that this scheme has much smaller time-truncation errors than those of standard methods.

K. Altiparmak [2] in his work, presented the Economized-rational approximations to find numerical solution of the one-dimensional Burgers' equation by discretizing the region in space over which the P.D.E.s are to be integrated using some mesh or grid, following which the space derivatives in the P.D.E.s and in the boundary conditions are replaced by some approximants.

D. S. Zhang, G.W. Wei and D.J. Kouri [7] numerically solved the Burgers' equation involving very high Reynolds numbers. They used a new approach based on the distributed approximating functional for representing spatial derivatives of the velocity field. They discovered that for moderately large Reynolds numbers, this simple approach can provide very high accuracy while using a small number of grid points. In the case where the Reynolds numbers $\geq 10^5$, the exact solution is not available and the literature had a lot of discrepancy.

Lyle C. Smith [13] did a numerical study of the viscous Burgers' equation with Robin's boundary conditions. He developed and tested two separate finite element and Galerkin schemes.

He showed that the Galerkin/Conservation method give better results than the Galerkin method. He treated the Burgers' equation as a perturbation of the linear heat equation with the appropriate realistic constants.

In this paper, we use mixed Galerkin finite element method to investigate the effect of a forcing term on the solutions of a homogeneous Burgers' equation with Robin's boundary conditions on the interval

[0, L]. We consider the Homogeneous Burgers' equation

$$\frac{\partial w}{\partial t}(t,x) + w(t,x)\frac{\partial w}{\partial x}(t,x) = \varepsilon \frac{\partial^2 w}{\partial x^2}(t,x) \qquad (1.1.7)$$

With Robin's boundary conditions

$$\frac{K_1}{L_1} w(t,0) - K\frac{\partial w}{\partial x}(t,0) = \frac{K_1}{L_1} u_1(t) \qquad (1.1.8)$$

$$\frac{K_2}{L_2} w(t,L) - K\frac{\partial w}{\partial x}(t,L) = \frac{K_2}{L_2} u_2(t) \qquad (7)$$

And initial condition

$$w(0,x) = \phi(x) \qquad (8)$$

II. GALERKIN METHOD WITH ROBINS BOUNDARY CONDITIONS

Now our problem takes the form

$$\dot{\mathbf{d}}(t) = -[Y]\mathbf{P}(\mathbf{d}(t)) + \varepsilon [B]\mathbf{u}(t) + \varepsilon [A]\mathbf{d}(t) \qquad (9)$$

Given the initial condition $\mathbf{d_0} = [Y]^{-1}\mathbf{w_0}$ which yields the matrix

$$
\begin{bmatrix} \dot{d}_0(t) \\ \dot{d}_1(t) \\ . \\ . \\ . \\ \dot{d}_{N+1}(t) \end{bmatrix}_{(N+2)\times 1} = \frac{-h}{36} \begin{bmatrix} 2 & 1 & 0 & 0 & .... & 0 \\ 1 & 4 & 1 & 0 & .... & 0 \\ 0 & 1 & 4 & 1 & .... & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & .... & 1 & 4 & 1 \\ 0 & 0 & .... & 0 & 1 & 2 \end{bmatrix}^{-1}_{(N+2)\times(N+2\}}
$$

$$
\begin{bmatrix} -2(d_0(t))^2 + d_o(t)d_1(t) + (d_1(t))^2 \\ -(d_0(t))^2 + d_o(t)d_1(t) + d_1(t)d_2(t) + (d_2(t))^2 \\ . \\ . \\ . \\ -(d_{N+1}(t))^2 + d_{N+1}(t)d_N(t) + d_N(t)d_{N+1}(t) + (d_{N=1}(t))^2 \\ -(d_N(t))^2 + d_N(t)d_{N+1}(t) + 2(d_{N+11}(t))^2 \end{bmatrix}_{(N+2)\times 1}
$$

$$
+\frac{\varepsilon h}{6} \begin{bmatrix} 2 & 1 & 0 & 0 & .... & 0 \\ 1 & 4 & 1 & 0 & .... & 0 \\ 0 & 1 & 4 & 1 & .... & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & .... & 1 & 4 & 1 \\ 0 & 0 & .... & 0 & 1 & 2 \end{bmatrix}^{-1}_{(N+2)\times(N+2)} \begin{bmatrix} \frac{k_1}{kL_1} & 0 \\ 0 & 0 \\ . & . \\ . & . \\ . & . \\ 0 & \frac{k_2}{kL_2} \end{bmatrix}_{(N+2)\times 2} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}_{2\times 1}
$$

$$
+\frac{\varepsilon}{6} \begin{bmatrix} 2 & 1 & 0 & 0 & .... & 0 \\ 1 & 4 & 1 & 0 & .... & 0 \\ 0 & 1 & 4 & 1 & .... & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & .... & 1 & 4 & 1 \\ 0 & 0 & .... & 0 & 1 & 2 \end{bmatrix}^{-1}_{(N+2)\times(N+2)}
$$

$$\begin{bmatrix} -1-\frac{hk_1}{kL_1} & 1 & 0 & 0 & .... & 0 \\ 1 & -2 & 1 & 0 & .... & 0 \\ 0 & 1 & -2 & 1 & .... & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & .... & 1 & -2 & 1 \\ 0 & 0 & .... & 0 & 1 & -1-\frac{hk_2}{kL_2} \end{bmatrix}_{(N+2)\times(N+2)} \begin{bmatrix} d_0(t) \\ d_1(t) \\ . \\ . \\ . \\ d_{N+1}(t) \end{bmatrix}_{(N+2)\times 1}$$

With initial conditions $\mathbf{d_0} = [Y]^{-1}\mathbf{w_0}$

$$+\varepsilon \begin{bmatrix} \frac{k_1}{kL_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ . & . \\ . & . \\ . & . \\ \mathbf{0} & \frac{k_2}{kL_2} \end{bmatrix}_{(N+2)\times 2} \qquad (11)$$

VI. RESULTS

Problems 9 is set up in MATLAB and solved using the ordinary differential equation solver. We employ numerical approximations using N=16 elements. This is the most cost effective number elements [2]. For an initial condition

$w_0(x) = \frac{1}{4}\cos(\pi x + t)$ And $\varepsilon = \frac{1}{60}$ let's assume a copper rod of length 1 metre and thermal conductivity 0.93 calories/second.centimetre.degrees Celsius having two films on both sides of length 10 cm each and thermal conductivity of 0.55 calories/second.centimetre.degrees Celsius. Setting the controls $u_1(t)$ and $u_2(t)$ at zero and assuming an exact solution of the form $\frac{\mathbf{1}}{\mathbf{4}}e^{-\varepsilon t}\cos(\pi x + t)$ at a final $t = \frac{1}{2}$ our results are obtained in the following tables.

Results for $\varepsilon = \frac{1}{60}$

Table 1: Approximate finite element solutions

x	Galerkin with Robins Boundary Conditions	Solutions with a forcing term
0	0.2499	0.2474
0.0588	0.2452	0.2432
0.1176	0.2326	0.2307
0.1765	0.2113	0.2103
0.2353	0.1834	0.1827
0.2941	0.1493	0.1489
0.3529	0.1105	0.1100
0.4118	0.0674	0.0673
0.4706	0.0228	0.02292
0.5294	-0.0228	-0.0223
0.5882	-0.0674	-0.0674
0.6471	-0.1105	-0.1102
0.7059	-0.1493	-0.1489
0.7647	-0.1834	-0.1827
0.8235	-0.2112	-0.2104
0.8824	-0.2324	-0.2307
0.9412	-0.2477	-0.2433
1	-0.2499	-0.2425

Results for $\varepsilon = \frac{1}{120}$

Table 2: Approximate finite element solutions

x	Galerkin with Robins Boundary Conditions	Solutions with a forcing term
0	0.2560	0.2485
0.0588	0.2489	0.2443
0.1176	0.2361	0.2316
0.1765	0.2139	0.2113
0.2353	0.1860	0.1837
0.2941	0.1527	0.1470
0.3529	0.1122	0.1060
0.4118	0.0685	0.06770
0.4706	0.0236	0.0200
0.5294	-0.0237	-0.0200
0.5882	-0.0685	-0.0019
0.6471	-0.1122	-0.1070
0.7059	-0.1527	-0.1450
0.7647	-0.1860	-0.1837
0.8235	-0.2139	-0.2121
0.8824	-0.2361	-0.2317
0.9412	-0.2489	-0.2445
1	-0.2560	-0.2476

VII. CONCLUSION

The numerical results obtained in this paper show that the forcing term converges the solutions of the homogeneous burgers equation initially but as the values of x increase the solutions to the homogeneous equation begin to diverge. It's notably observed that at some point the forcing term produces almost constant solutions but on minimal occasions.but important to note is the fact that in all cases though the solutions observe some steady state behaviour.

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