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# Unit Groups of Some Classes of Power Four Radical Zero Commutative Completely Primary Finite Rings

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**Abstract**

Let $R$ be a completely primary finite ring and $J$ be its Jacobson radical. A class of such rings in which $J^4 = (0)$, $J^3 \neq (0)$ has been constructed. Moreover, the structures of their groups of units have been determined for all the characteristics of $R$.

**Mathematcs Subject Classification:** Primary 13M05, 16P10, 16U60; Secondary 13E10, 16N20

**Keywords:** Unit Groups, Completely Primary Finite Rings

## 1 Introduction

The background information on completely primary finite rings can be obtained from [1, 5]. The notations used in this paper are standard, see e.g [5].

The classification of finite rings, still remains open. Several authors have constructed finite rings whose Jacobson radical or group of units yield a particular structure. For instance, in [2], the author has obtained structures of unit groups of classes of completely primary finite rings in which the product of any three zero divisors is zero. It is well known that if $R$ is a finite field, then the group of units is cyclic. In [3], Gilmer characterized all rings whose groups of units are cyclic. In this paper, we have constructed rings in which the product of any four zero divisors is zero. Moreover, their groups of units have been characterized. From the well known Raghavendran's result (see[5],Theorem 2), we notice that if $J^4 = (0), J^3 \neq (0)$, then characteristic of $R$ is $p$, $p^2$, $p^3$ and $p^4$.

## 2 Rings of characteristic $p$

Let $R^{'} = GR(p^r, p)$ be the Galois ring of order $p^r$ and characteristic $p$. Suppose $U$, $V$ and $W$ are finitely generated $R^{'}$ modules such that $\dim_{R^{'}} U = s$, $\dim_{R^{'}} V = t$, $\dim_{R^{'}} W = \lambda$ and $s + t + \lambda = h$. Let $\{u_1, ..., u_s\}, \{v_1, ..., v_t\}$ and $\{w_1, ..., w_\lambda\}$ be the generators of $U$, $V$ and $W$ respectively so that $R = R^{'} \oplus U \oplus V \oplus W$ is an additive abelian group. Further, assume that $s = 1, t = 1, \lambda = h-2$ so that $R = R^{'} \oplus R^{'}u \oplus R^{'}v \oplus \sum_{j=1}^{h-2} R^{'}w_j$ and $pu = pv = pw_j = 0, 1 \leq j \leq h-2$. On $R$, define multiplication as follows:
$(r_0, r_1, r_2, ..., r_h)(s_0, s_1, ..., s_h) = (r_0s_0, r_0s_1 + r_1s_0, r_0s_2 + r_2s_0 + r_1s_1, r_0s_3 + r_3s_0 + r_1s_2 + r_2s_1, ..., r_0s_h + r_hs_0 + r_1s_2 + r_2s_1)$. It is easy to verify that the given multiplication turns $R$ into a commutative ring with identity $(1, 0, ..., 0)$.

**Proposition 1.** *The ring constructed in this section is completely primary of characteristic $p$ and*

$$J = R^{'}u \oplus R^{'}v \oplus \sum_{j=1}^{\lambda} R^{'}w_j$$

$$J^2 = R^{'}v \oplus \sum_{j=1}^{\lambda} R^{'}w_j$$

$$J^3 = \sum_{j=1}^{\lambda} R^{'}w_j$$

$$J^4 = (0)$$

**Proposition 2.** *see e.g [2]. Let $R$ be a completely primary finite ring (not necessarily commutative). Then, the group of units, $R^*$ of the ring $R$ contains a cyclic subgroup $< b >$ of order $p^r - 1$ and $R^*$ is a semi direct product of $1 + J$ and $< b >$.*

**Proposition 3.** *Let $R$ be the ring constructed in this section, and $J$ be its Jacobson radical. Then*

$$R^* \cong \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_p^r)^\lambda \text{ if } p \neq 2$$

*for every prime integer $p$ and positive integer $r$.*

*Proof.* Let $\xi_1, ..., \xi_r \in R'$ with $\xi_1 = 1$ such that $\overline{\xi_1}, ..., \overline{\xi_r} \in R'/pR'$ form a basis for $R'/pR'$ regarded as a vector space over its prime subfield $F_p$.
For every prime integer $p$, $1 + J$ is a direct product of the following $r(h-1)$ cyclic subgroups with their respective orders

$$\{(1 + \xi_\nu u)^{\alpha_\nu} \mid 1 \leq \alpha_\nu \leq p^2\}, 1 \leq \nu \leq r$$
$$\{(1 + \xi_\nu w_j)^{\beta_\nu} \mid 1 \leq \beta_\nu \leq p\}, 1 \leq \nu \leq r, 1 \leq j \leq \lambda$$

The structure of $R^*$ follows from the above Proposition 2. □

# 3 Rings of characteristic $p^2$

Let $R' = GR(p^{2r}, p^2)$ be the Galois ring of order $p^{2r}$ and characteristic $p^2$. Suppose $U$, $V$ and $W$ are finitely generated $R'$ modules such that $\dim_{R'} U = s$, $\dim_{R'} V = t$, $\dim_{R'} W = \lambda$ and $s + t + \lambda = h$. Let $\{u_1, ..., u_s\}, \{v_1, ..., v_t\}$ and $\{w_1, ..., w_\lambda\}$ be the generators of $U$, $V$ and $W$ respectively so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group. Further, assume that $s = h-1, t = 1, \lambda = 0$ so that $R = R' \oplus \sum_{j=1}^{h-1} R' u_j \oplus R' v$ where $pu_j \neq 0$, $p^2 u_j = 0, 1 \leq j \leq s$ and $pv = 0$. On $R$, define multiplication as follows:
$(r_0, r_1, r_2, ..., r_{h-1}, \overline{r_h})(s_0, s_1, ..., s_{h-1}, \overline{s_h}) = (r_0 s_0 + p \sum_{i,j=1}^{h-1} r_i s_j, r_0 s_1 + r_1 s_0, ..., r_0 s_{h-1} + r_{h-1} s_0, r_0 \overline{s_h} + \overline{r_h} s_0)$ where $\overline{r_h}, \overline{s_h} \in R'/pR'$.It is easy to verify that the given multiplication turns $R$ into a commutative ring with identity $(1, 0, ..., 0, \overline{0})$.

**Proposition 4.** *The ring constructed in this section is completely primary of characteristic $p^2$ and*

$$J = pR' \oplus \sum_{j=1}^{s} R' u_j \oplus R' v$$

$$J^2 = pR' \oplus p \sum_{j=1}^{s} R' u_j \oplus R' v$$

$$J^3 = p \sum_{j=1}^{s} R' u_j$$

$$J^4 = (0).$$

**Proposition 5.** *Let $R$ be the ring constructed in this section, and $J$ be its Jacobson radical. Then*

$$R^* \cong \mathbb{Z}_{p^r-1} \times (\mathbb{Z}^r_{p^2})^s \times (\mathbb{Z}^r_p)^2$$

*for every prime integer $p$ and positive integer $r$.*

*Proof.* Let $\xi_1, ..., \xi_r \in R^{'}$ with $\xi_1 = 1$ such that $\overline{\xi_1}, ..., \overline{\xi_r} \in R^{'}/pR^{'}$ form a basis for $R^{'}/pR^{'}$ regarded as a vector space over its prime subfield $F_p$. It therefore suffices to characterize $1+J$, for the structure of $R^*$ will easily follow from Proposition 2.
For every prime integer $p$, $1 + J$ is a direct product of the following $r(s + 2)$ cyclic subgroups with their respective orders

$$\{(1 + p\xi_\nu)^\alpha\} \mid 1 \leq \alpha \leq p\}, 1 \leq \nu \leq r$$
$$\{(1 + \xi_\nu u_j)^{\beta_j} \mid 1 \leq \beta_j \leq p^2\}, 1 \leq \nu \leq r, 1 \leq j \leq s$$
$$\{(1 + p\xi_\nu v)^{k_\nu} \mid 1 \leq k_\nu \leq p\}, 1 \leq \nu \leq r$$

The structure of $R^*$ follows from the above Proposition 2. □

# 4 Rings of characteristic $p^3$

Let $R^{'} = GR(p^{3r}, p^3)$ be the Galois ring of order $p^{3r}$ and characteristic $p^3$. Suppose $U$, $V$ and $W$ are finitely generated $R^{'}$ modules such that $\dim_{R^{'}} U = s$, $\dim_{R^{'}} V = t$, $\dim_{R^{'}} W = \lambda$ and $s + t + \lambda = h$. Let $\{u_1, ..., u_s\}, \{v_1, ..., v_t\}$ and $\{w_1, ..., w_\lambda\}$ be the generators of $U$, $V$ and $W$ respectively so that $R = R^{'} \oplus U \oplus V \oplus W$ is an additive abelian group. Further, assume that $s = h-1, t = 1, \lambda = 0$ so that $R = R^{'} \oplus \sum_{j=1}^{h-1} R^{'} u_j \oplus R^{'} v$ where $p^2 u_j \neq 0, p^3 u_j = 0, 1 \leq j \leq s$ and $pv = 0$. On $R$, define multiplication as follows:
$(r_0, \overline{r_1}, \overline{r_2}, ..., \overline{r_{h-1}}, \widetilde{r_h})(s_0, \overline{s_1}, ..., \overline{s_{h-1}}, \widetilde{s_h}) = (r_0 s_0, r_0 \overline{s_1} + \overline{r_1} s_0, ..., r_0 \overline{s_{h-1}} + \overline{r_{h-1}} s_0,$ $r_0 \widetilde{s_h} + \widetilde{r_h} s_0 + \sum_{i,j=1}^{h-1} \overline{r_i s_j})$ where $\overline{r_i}, \overline{s_j} \in R^{'}/p^2 R^{'}$ and $\widetilde{r_h}, \widetilde{s_h} \in R^{'}/pR^{'}$.It is readily verified that the given multiplication turns $R$ into a commutative ring with identity $(1, \overline{0}, ..., \overline{0}, \widetilde{0})$.

**Proposition 6.** *The ring constructed in this section is completely primary of characteristic $p^3$ and*

$$J = pR^{'} \oplus \sum_{j=1}^{s} R^{'} u_j \oplus R^{'} v$$

$$J^2 = p^2 R^{'} \oplus p \sum_{j=1}^{s} R^{'} u_j \oplus R^{'} v$$

$$J^3 = pR^{'}v$$
$$J^4 = (0).$$

**Proposition 7.** *Let $R$ be the ring constructed in this section, and $J$ be its Jacobson radical. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4^{r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_4^r \times (\mathbb{Z}_2^r)^{s-1} \text{ if } p = 2 \\ \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^2}^r)^s \text{ if } p \neq 2 \end{cases}$$

*Proof.* Let $\xi_1, ..., \xi_r \in R^{'}$ with $\xi_1 = 1$ such that $\overline{\xi_1}, ..., \overline{\xi_r} \in R^{'}/pR^{'}$ form a basis for $R^{'}/pR^{'}$ regarded as a vector space over its prime subfield $F_p$.
If $p = 2$, consider $\nu = 1, ..., r$ and $y \in R^{'}$ such that $x^2 + x + \overline{y} = \overline{0}$ over $R^{'}/pR^{'}$ has no solution in the field $R^{'}/pR^{'}$. We easily notice that $1 + J$ is a direct product of the following $r(s + 2) + 1$ cyclic subgroups with their respective orders.

$$\{(-1 + 4\xi_1)^\alpha \mid \alpha = 1, 2\}$$
$$\{(1 + 4y)^\beta \mid \beta = 1, 2\}$$
$$\{(1 + 2\xi_\nu)^{\kappa_\nu} \mid \kappa_\nu = 1, ..., 4\}, 2 \leq \nu \leq r$$
$$\{(1 + 2\xi_\nu u_j)^{\tau_{j_\nu}} \mid \tau_{j_\nu} = 1, 2\}, \nu = 1, ..., r; j = 1, ..., s - 1$$
$$\{(1 + \xi_\nu u_{s-1} + \xi_\nu u_s)^{\gamma_\nu} \mid \gamma_\nu = 1, ..., 4\}, \nu = 1, ..., r$$
$$\{(1 + \xi_\nu u_s + \xi_\nu v)^{\delta_\nu} \mid \delta_\nu = 1, ..., 8\}, \nu = 1, ..., r$$

If $p \neq 2$, $1 + J$ is a direct product of the following $r(s + 2)$ cyclic subgroups with their respective orders:

$$\{(1 + p\xi_\nu)^{\alpha_\nu} \mid \alpha_\nu = 1, ..., p^2\}, \nu = 1, ..., r$$
$$\{(1 + \xi_\nu u_j)^{\beta_{j_\nu}} \mid \beta_{j_\nu} = 1, ..., p^2\}, j = 1, ..., s; \nu = 1..., r$$
$$\{(1 + \xi_\nu v)^{\alpha_\nu} \mid \alpha_\nu = 1, ..., p^2\}, \nu = 1, ..., r.$$

In both cases, the structure of $R^*$ follows from Proposition 2. □

# 5 Rings of characteristic $p^4$

Let $R^{'} = GR(p^{4r}, p^4)$ be the Galois ring of order $p^{4r}$ and characteristic $p^4$. Suppose $U$, $V$ and $W$ are finitely generated $R^{'}$ modules such that $\dim_{R^{'}} U = s$, $\dim_{R^{'}} V = t$, $\dim_{R^{'}} W = \lambda$ and $s + t + \lambda = h$. Let $\{u_1, ..., u_s\}, \{v_1, ..., v_t\}$ and $\{w_1, ..., w_\lambda\}$ be the generators of $U$, $V$ and $W$ respectively so that $R = R^{'} \oplus U \oplus V \oplus W$ is an additive abelian group. Further, assume that $s = h, t = 0, \lambda = 0$ so

that $R = R^{'} \oplus \sum_{j=1}^{s} R^{'} u_j$ where $pu_j = 0, 1 \leq j \leq s$. On $R$ define multiplication as follows:
$(r_0, \overline{r_1}, \overline{r_2}, ..., \overline{r_h})(s_0, \overline{s_1}, ..., \overline{s_h}) = (r_0 s_0, r_0 \overline{s_1} + \overline{r_1} s_0, ..., r_0 \overline{s_h} + \overline{r_h} s_0)$ where $\overline{r_i}, \overline{s_j} \in R^{'}/pR^{'}$, $1 \leq i, j \leq s$. This multiplication turns $R$ into a commutative ring with identity $(1, \overline{0}, ..., \overline{0})$.

**Proposition 8.** *The ring constructed in this section is completely primary of characteristic $p^4$ and*

$$J = pR^{'} \oplus \sum_{j=1}^{s} R^{'} u_j$$

$$J^2 = p^2 R^{'}$$

$$J^3 = p^3 R^{'}$$

$$J^4 = (0).$$

**Proposition 9.** *Let $R$ be the ring constructed in this section, and $J$ be its Jacobson radical. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8^{r-1} \times (\mathbb{Z}_2^r)^s \text{ if } p = 2 \\ \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_p^r)^s \text{ if } p \neq 2 \end{cases}$$

*Proof.* See Proposition 5 in [4]. □

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