**Pure Mathematical Sciences, Vol. 4, 2015, no. 3, 121 - 126**
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**http://dx.doi.org/10.12988/pms.2015.556**

# On the Quotient Groups of Subgroups of the Unit Groups of a Class of Completely Primary Finite Rings

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**Abstract**

The study of completely primary finite rings has generated interesting results in the structure theory of finite rings with identity. It has been shown that a finite ring can be classified by studying the structures of its group of units. But this group has subgroups which are interesting objects of study. Let $R$ be a completely primary finite ring of characteristic $p^n$ and $J$ be its Jacobson radical satisfying the condition $J^n = (0)$ and $J^{n-1} \neq (0)$. In this paper, we characterize the quotient groups of subgroups of the group of units of $R$.

**Mathematics Subject Classification:** Primary 13M05, 16P10, 16U60, Secondary 13E10, 16N20

**Keywords:** Completely primary finite rings, unit groups and quotient groups.

## 1. Introduction

Research on finite rings and their applications has produced admirable results in abstract algebra. A completely primary finite ring is a finite ring $R$ with identity $1 \neq 0$ whose subset of all the zero divisors forms a unique maximal ideal $J$. Although the classification of commutative completely primary finite rings has produced good research material in the recent past, the characterization of their groups of units has not been exhaustively done. In this paper, the quotient groups of the subgroups of the unit groups studied in [3] have been characterized. For the most recent related work and notations, reference can be made to [2], [3] and [4].

## 2. The construction

Let $R_0$ be the Galois ring of the form $GR(p^{nr}, p^n)$. For each $i = 1, ..., h$, let $u_i \in J(R_0)$ such that $U$ is an $h$ dimensional $R_0-$ module generated by $u_1, ..., u_h$ so that $R_0 \oplus U$ is an additive group. On this group, define multiplication by the following relations:

(i)If $n = 1, 2$ then $pu_i = u_iu_j = u_ju_i = 0$, $u_ir_0 = r_0u_i$.

(ii)If $n > 3$ then $p^{n-1}u_i = 0$, $u_iu_j = p^2\gamma_{ij}, {u_i}^n = {u_i}^{n-1}u_j = u_i{u_j}^{n-1} = 0, u_ir_0 = r_0u_i$, where $r_0, \gamma_{ij} \in R_0$, $1 \leq i, j \leq h$, $p$ is a prime integer, $n$ and $r$ are positive integers. Moreover if $u_i|_U$, then the additive order of $u_i$ is $p$.

*Remark* 1. If $n = 1$ or 2, then the construction yields rings in which multiplication of any two zero divisors is zero, that is $J^2 = (0)$. Such rings have been shown to be completely primary and their group of units is well known. Reference can be made to [3].

*Remark* 2. From the above construction, we see that every element of $R$ may uniquely be expressed as $r = r_0 + r_1u_1 + r_2u_2 + ... + r_hu_h$ for $r_i \in R_0$, $u_j \in U$, $0 \leq i \leq h$, $1 \leq j \leq h$.

*Remark* 3. The constructed ring is known to be completely primary and can be classified under rings satisfying the conditions $J^n = (0)$, $J^{n-1} \neq (0)$.

**Proposition 1.** *Let $R$ be the ring constructed above. Then*

*(i)* $J = pR_0 \oplus U$.

*(ii)* $J^{n-1} = p^{n-1}R_0$.

*(iii)*$J^n = (0)$.

*See e.g [3].*

## 3. Preliminary Results

### 3.1 The group of units

The following theorems describe the structure of the multiplicative group $R_0{}^*$ for any Galois ring $R_0$ and the constructed ring $R$ respectively.

**Theorem 1.** *Let $R_0{}^*$ be the multiplicative group of the invertible elements of the ring $GR(p^{nr}, p^n)$ where $p$ is a prime. Then $R_0{}^*$ is the direct product of the cyclic group $< a >$ of order $p^r - 1$ and the unit group $1 + J$ of order $p^{(n-1)r}$. See e.g [2].*

**Theorem 2.** *The unit group $R^*$ of the commutative completely primary finite ring $R$ of characteristic $p^n$ with maximal ideal $J$ such that $J^2 = (0)$ when $n = 1, 2$ and $J^n = (0)$, $J^{n-1} \neq (0)$ when $n \geq 3$ and with invariants $p$ (prime integer) $p \in J$, $r \geq 1$ and $n \geq 1$ is a direct product of cyclic groups as follows:*
*(i)If char $R = p$ then*

$$R^* \cong \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^h.$$

*(ii)If char $R = p^2$ then*

$$R^* \cong \mathbb{Z}_{p^r-1} \times \mathbb{Z}_p^r \times (\mathbb{Z}_p^r)^h.$$

*(iii)If char $R = p^n$ for $n \geq 3$, then*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^{n-1}}^{r-1} \times (\mathbb{Z}_2^r)^h. & \text{if } p = 2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^{n-1}}^r \times (\mathbb{Z}_p^r)^h. & \text{if } p \neq 2. \end{cases}$$

*Proof.* See [3]. □

### 3.2 The quotient groups

Let $R$ be the commutative finite ring constructed in section 2 above. Notice that $J^n = (0)$ and $J^{n-1} \neq (0)$ with char $R{=}p^n$. Now $R$ is of order $p^{(n+h)r}$ and the residue field $R/J$ is the finite field $GF(p^r)$ for some prime integer $p$ and positive integer $r$. Since $R$ is of order $p^{(n+h)r}$ then $|R^*|{=}|R-J|{=}(p^r-1)p^{(n-1)r+rh}$ and $|1+J| = p^{(n-1)r+rh}$ is an abelian $p-group$. It is also clear that $1 + J$ is a normal subgroup of its group of units $R^*$. If we let $n \geq 2$ then the ideals $J, J^2, J^3, ..., J^{n-1}$ and $J^n$ in $R$ form a chain $J \supset J^2 \supset J^3 \supset ... \supset J^n = (0)$ and consequently the subgroups $1 + J, 1 + J^2, 1 + J^3, ..., 1 + J^n = \{1\}$ form a chain $1 + J \supset 1 + J^2 \supset 1 + J^3 \supset ... \supset 1 + J^{n-1} \supset 1 + J^n = \{1\}$. Given the fact that $1 + J^i$ for $i = 1, ..., n$ form the normal subgroups of the unit group $R^*$, we can express the unit group $R^*$ as follows:
$R^* \cong (R^*/1 + J) \times (1 + J)$. It is seen that the expression is a direct product of two subgroups. The subgroup $(R^*/1 + J)$ which is cyclic of order $p^r - 1$ isomorphic to $\mathbb{Z}_{p^r-1}$. The aim of this paper is to determine the structures of

quotient groups of the subgroups of $1+J$.
Now, $1+J$ is an abelian $p-$subgroup of the unit group $R^*$. The group $1+J$ has a normal series $\{1\} = 1+J^n \trianglelefteq 1+J^{n-1} \trianglelefteq ... \trianglelefteq 1+J^2 \trianglelefteq 1+J$ with the factor groups $(1+J/1+J^2), (1+J^2/1+J^3)...(1+J^{n-1}/1+J^n)$ isomorphic to the additive groups $J/J^2, J^2/J^3...J^{n-1}$. When $n \geq 3$ then $1+J$ is not a composition series since $1+J^2$ is not maximal in $1+J$, so that $1+J/1+J^2$ is not simple. By Lagrange's theorem
$|1+J| = |(1+J/1+J^2)|.|(1+J^2/1+J^3)|...|(1+J^{n-1}/1+J^n)| = p^{(n-1)r+rh}$ for $n \geq 2$ .
We therefore give the structure of $(1+J/1+J^2)$ and $(1+J^l/1+J^m)$ for $m \geq 3$ and $2 \leq l < m$.

**Proposition 2.** *Let $1 \leq i < l$, then $1+J^l \trianglelefteq 1+J^i$.*

*Proof.* It suffices to prove that $1+J^l$ is a subgroup of $1+J^i$.
Let $1+p^l r_0, 1+p^l s_0 \in 1+J^l$. Then $(1+p^l s_0)^{-l} = 1-p^l s_0 + p^{2l}s_0^2 - p^{3l}s_0^3 + ... + (-1)^{n+1}p^{l(n-1)}s_0^{n-1}$ so that
$(1+p^l r_0)(1+p^l s_0)^{-1} = (1+p^l r_0)(1-p^l s_0+p^{2l}s_0^2-p^{3l}s_0^3+....+(-1)^{n+1}p^{l(n-1)}s^{n-1})$
$= 1+p^l((r_0-s_0)+p^l(s_0^2-r_0s_0)+...+(-1)^{n+1}p^{l(n-2)}(s_0^{n-1}-r_0s^{n-2})) \in 1+J^l$. □

The following results can be easily established.

**Proposition 3.** *Let $l, s \in \mathbb{Z}^+ \ \{1\}$, $1+J^l$ and $1+J^s$ be subgroups of $1+J$. Suppose $\theta : 1+J^l \to 1+J^s$ is a homomorphism, then kernel of $\theta$ is a normal subgroup of $1+J^l$.*

**Proposition 4.** *Let $R$ be the ring given by the construction.*
*i) When $r=1$ and $n \geq 3$ then $1+J/1+J^2 \cong (\mathbb{Z}_p)^{h+1}$ and $1+J^l/1+J^{l+1} \cong \mathbb{Z}_p$ for $2 \leq l \leq n-1$.*
*ii) If $1 \leq l \leq m \leq t \leq n$, then the $p-$ group $1+J$ is a direct product of the subgroups $1+p^lR_0$ by $1+\sum_{i=1}^{h} \oplus R_0/pR_0$ and*

$$(1+J^l/1+J^m) \cong (1+J^l/1+J^t)/(1+J^m/1+J^t)$$

**Corollary 1.** *Let $1 \leq i < s \leq n$. Then*

$$R^*/1+J^i \cong (R^*/1+J^s)/(1+J^i/1+J^s)$$

From the constructed ring in section 2, the quotient $J^{n-1}/J^n$ is a vector space over the ring $R/J \cong \mathbb{F}_p$.

**Lemma 1.** *Let $J$ be the Jacobson radical of the ring $R$ constructed in section 2, then the quotient $J^{n-1}/J^n$ for $n \geq 2$ is a vector space over $GF(p) \subseteq R/J$. see [2].*

*Proof.* Given that $J$ is a maximal ideal in $R$, the quotient ring $R/J$ is a field. For any prime integer $p$, let $\mathbb{F}_p$ be a prime subfield of $R/J$. Let $y_1, y_2 \in J^{n-1}$ such that $y_1 + J^n$ and $y_2 + J^n$ belong to $J^{n-1}/J^n$, then for each $a \in \mathbb{F}_p$ we have that $a((y_1+J^n)+(y_2+J^n)) = a((y_1+y_2)+J^n) = (a(y_1+y_2))+J^n$ which belongs to $J^{n-1}/J^n$. Now $|R| = |R/J|.|J/J^2|, ..., |J^{n-1}/J^n|.|J^n| = p^{nr+rh}$. Thus $R$ is indeed finite. □

*Remark* 4. Finiteness of $R$ implies that $J$ is nilpotent say $J^n = (0)$.

## 4. Main results

**Theorem 3.** *Let $R$ be the ring constructed in section 2. Suppose $J$ is the Jacobson radical of $R$, then the quotient group $1 + J/1 + J^2 \cong (\mathbb{Z}_p^r)^{h+1}$ for every prime integer $p$ and positive integer $r$.*

*Proof.* Let $\tau_1, ..., \tau_r \in R_0$ such that $\overline{\tau_1}, ..., \overline{\tau_r} \in R_0/pR_0$ form a basis for $R_0/pR_0$ regarded as a vector space over its prime subfield $F_p$.
For $\nu = 1, ..., r$ , consider the element $(1 + p\tau_\nu)(1 + J^2) \in (1 + J/1 + J^2)$.
Then $((1+p\tau_\nu)(1+J^2)^p = (1+p\tau_\nu)^p(1+J^2) =(1+p^2\tau_\nu+, ..., +p^p\tau_\nu^p)(1+J^2) = 1 + J^2$ since characteristic of $R = p^2$.
Next consider the element $(1 + \tau_\nu u_1)(1 + J^2) \in (1 + J/1 + J^2)$.
Then $((1+\tau_\nu u_1)(1+J^2)^p = (1+\tau_\nu u_1)^p(1+J^2) = 1+J^2$ since $1+p\tau_\nu u_1 \in (1+J^2)$.
Next, the element $((1 + \tau_\nu u_2)(1 + J^2)^p = (1 + J^2)$.
Continuing in a similar manner up to the element $(1+\tau_\nu u_h)(1+J^2)$ we obtain $((1 + \tau_\nu u_h)(1 + J^2)^p = (1 + J^2)$. If we let

$$T_\nu = \{((1 + p\tau_\nu)1 + J^2)^a | a = 1, ..., p\}$$

$$S_{1\nu} = \{((1 + \tau_\nu u_1)1 + J^2)^{b_1} | b_1 = 1, ..., p\}$$
$$S_{2\nu} = \{((1 + \tau_\nu u_2)1 + J^2)^{b_2} | b_2 = 1, ..., p\}$$
$$\vdots$$
$$S_{h\nu} = \{((1 + \tau_\nu u_h)1 + J^2)^{b_h} | b_h = 1, ..., p\},$$

we see that they are all subgroups of the group $(1 + J/1 + J^2)$ and they are of the orders indicated by their definition. Since
$\prod_{\nu=1}^r | < (1 + p\tau_\nu)(1 + J^2) > |.\prod_{\nu=1}^r \prod_{i=1}^h | < (1 + \tau_\nu u_i)(1 + J^2) > | = p^{r(h+1)}$
and the intersection of any pair of the cyclic subgroups gives the identity group $(1 + J^2)$, the product of the $(h + 1)r$ subgroups $T_\nu, S_{1\nu}, S_{2\nu}, ..., S_{h\nu}$ is direct so their product exhausts the group $(1 + J/1 + J^2)$. □

**Theorem 4.** *Let $R$ be the ring constructed in section 2. Suppose $J$ is the Jacobson radical of $R$, then for $n \geq 3$, $2 \leq \ell < m$ the quotient group $(1 + J^\ell/1 + J^m) \cong \mathbb{Z}_{p^{m-\ell}}^r$ for every prime integer $p$ and positive integers $r, \ell, m$.*

*Proof.* Let $\tau_1, ..., \tau_r \in R_0$ such that $\overline{\tau_1}, ..., \overline{\tau_r} \in R_0/pR_0$ form a basis for $R_0/pR_0$ regarded as a vector space over its prime subfield $F_p$.
Consider the element $(1+p^\ell\tau_\nu)1+J^m \in (1+J^\ell/1+J^m)$ for $1 \leq \nu \leq r$. Then $((1+p^\ell\tau_\nu)1+J^m)^{p^{m-\ell}}= ((1+p^\ell\tau_\nu)^{p^{m-\ell}})1+J^m= (1+p^m\tau_\nu... + (p^\ell\tau_\nu)^{p^{m-\ell}})1+J^m = 1+J^m$, because $1+p^m\tau_\nu + ... + (p^\ell\tau_\nu)^{p^{m-\ell}} \in 1+J^m$. Now, $o((1+p^\ell\tau_\nu)1+J^m) = p^{m-\ell}$, since it is the smallest positive integer so that $((1+p^\ell\tau_\nu)1+J^m)^{p^{m-\ell}} = 1+J^m$. This assertion is easily verified by considering $((1+p^\ell\tau_\nu)1+J^m)^{p^{m-\ell-1}}=(1+p^{m-1}\tau_\nu + ... + (p^\ell\tau_\nu)^{p-m-1})1+J^m \neq 1+J^m$ because $1+p^{m-1}\tau_\nu + ... + (p^\ell\tau_\nu)^{p-m-1}$ is not an element of $1+J^m$. Therefore $1+J^\ell/1+J^m =< (1+p^\ell\tau_\nu)1+J^m > \cong \mathbb{Z}^r_{p^{m-\ell}}$. □

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**Received: May 23, 2015; Published: July 14, 2015**