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Classification of Ideals in Banach Spaces

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Authors' contributions

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Abstract

Let an operator T belong to an operator ideal J, then for any operators A and B which can be composed with T as BTA then $BTA \in J$. Indeed, J contains the class of finite rank Banach Space operators. Now given $L(X,Y)$. Then $J(X,Y) \subseteq L(X,Y)$ such that $J(X,Y) = \{T : X \longleftarrow Y : T \in J\}$. Thus an operator ideal is a subclass J of L containing every identity operator acting on a one-dimensional Banach space such that: $S + T \in J(X,Y)$ where $S, T \in J(X,Y)$. If $W, Z, X, Y \in \mathbb{K}, A \in L(W,X), B \in L(Y,Z)$ then $BTA \in J(W,Z)$ whenever $T \in J(X,Y)$. These properties compare very well with the algebraic notion of ideals in Banach Algebras within whose classes lie compact operators, weakly compact operators, finitely strictly regular operators, completely continuous operators, strictly singular operators among others. Thus, the aim of this paper is to characterize the various classes of ideals in Banach spaces. Special attention is given to the characteristics involving the ideal properties, the metric approximation properties, the hereditary properties in relation to the ideal extensions in the Hahn-Banach space, projection and embedments in the biduals of the Banach Space.

Keywords: Operator ideals; Banach space ideals.

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1 Introduction

Unless stated otherwise, the notations and symbols used in this paper are standard.

There are enormous applications of Ideals in Spectral Theory, Geometry of Banach spaces, Theory of eigenvalue distributions among others, that has necessitated their studies to occupy special importance in functional analysis. Many useful operator ideals have been defined by using sequence of $s-$numbers. For the unifications of different $s-$numbers, an analogy of axiomatic theory of $s-$numbers in Banach spaces is readily available in Literature (cf. [1]). Previously, Rhoades[2] generalized the classes of operators of l_p type and operators of Cesaro type by introducing an arbitrary infinite matrix $A = (a_{nk})$ using approximation numbers of a bounded linear operator. Alfsen and Efros[3] generalized the notion of two-sided ideals to Banach spaces, where they introduced M-ideals. The main idea was to generalize the two-sided ideals in a C^*algebra and obtained a variant which would serve as a tool for the study of Banach spaces. The notion of M-ideals is an appropriate generalization, since in a C^*-algebra, $M-$ideals coincide with the two-sided closed ideals[4].

Indeed, the subject of operator ideals and their characterization has been a subject of interest for quite some time now. Some of the advancements in this direction are attributed to Godefroy, Kalton, and Saphar [5]. In their work on unconditional ideals in Banach spaces they extended the notion of an ideal by relaxing some conditions and stated that if X is a subspace of a Banach space Y then X is said to be an ideal in Y if $X^\perp$ is the kernel of a contractive projection on Y^*. The study in [5] gives a general analogy of h-ideals and u-ideals. It is shown that if a separable Banach space X is an h-ideal in X^{**} then X has a complex form of Pelczynski's property (u) with constant one and the Baire-one function $Ba(X)$ in X^{**} are complemented by an Hermitian projection and the converse holding under a compatibility condition which is a necessity. This idea was related to the more familiar M-ideal and to the Banach lattices. Motivated by some ideas of Godun[6] they introduced the Godun set of a Banach space X denoted by $G(X)$ and defined as a set of all scalars λ such that $\|I - \lambda\pi\| = 1$ where π is the canonical projection of X^{***} onto X^*. If X contains a copy of l_1, then the Godun set $G(X)$ reduces to $\{0\}$. If X is separable and X^* is non separable then $G(X) \subset [0, 1]$. When X^* is separable and $1 < \lambda < 2$ it is shown that X can be renormed so that $[0, \lambda] \subset G(X)$. In this direction it is shown that a Banach space with separable dual can be renormed to satisfy hereditarily an "almost" optimal uniform smoothness condition. This optimal condition occurring when the canonical decomposition $X^{***} = X^\perp \oplus X^*$ is unconditional.

Rao's work in [7] on intersection of ideals in Banach space picked interest in studying finite intersections of ideals in Banach spaces. It is shown that for a Banach space X, if in the bidual X^{**}, every ideal of finite codimension is the intersection of ideals of codimension one, then the same property holds in X. The work further showed that a Banach space whose dual is isometric to $L^1(\mu)$ for a positive measure μ then any ideal of finite codimension is a finite intersection of ideals of codimension one. Moreover, Rao[8] introduced the notion of an extremely strict ideal. In particular, the study showed that the space of affine continuous functions on $\mathbb{K}$ is an extremely strict ideal in the space of continuous functions on $\mathbb{K}$. For injective tensor product spaces, a cancelation Theorem for extremely strict ideals was proved and non-reflexive Banach spaces which are not strict ideals in their fourth dual exhibited.

The study of Abrahamsen et al[9] on unconditional ideals of finite rank operators gave properties of $F(Y, X)$ as u-ideals in $W(Y, X)$ for every Banach spaces X and Y in terms of nets of finite rank operators approximating weakly compact operators. Similar characterizations were given for the cases when $F(Y, X)$ is a u-ideal in $W(Y, X)$ for every Banach space Y, when $F(Y, X)$ is a u-ideal in $W(Y, X)$ for every Banach space Y and when $F(Y, X)$ is a u-ideal in $K(Y, X)$ for every Banach space Y. Abrahamnsen et al[10] defined and studied λ-strict ideals in Banach spaces in which for $\lambda = 1$ means strict ideals. Strict u-ideals in their biduals are known to have the unique ideal property and the study in [10] revealed that the λ-strict u-ideals also have unique properties in their biduals, at least for $\lambda > 1/2$. Other studies closely related to this work can be found in [11, 12, 13, 14, 15, 16].

2 Preliminary Results

The characterization of Banach Algebra of compact operators as ideals can be demonstrated as follows:

Theorem 2.1. *[17] Let X be a separable Banach Algebra of compact operators. Then:*

i. $K(X, X^{**})$ *is an ideal in* $L(X, X^{**})$

ii. $K(X, X)$ *is an ideal in* $L(X, X)$

iii. $K(X, X)$ *is not an ideal in* $L(X, X^{**})$

Proof. Let $X = \left(\sum_{n=1}^{\infty} \oplus (Z^{**}, \|\cdot\|_n)\right)_2$ where Z^{**} is a separable Banach space and $\|\cdot\|_n$ is an equivalent norm on Z^{**}. The space X fails the metric compact approximation property, but its dual X^* has the metric approximation property. Since Z^{**} is separable, the space $(Z^{**}, \|\cdot\|_n)$ has the Radon-Nikodym property. Thus X has the Radon-Nikodym property as well (the fact that the Radon- Nikodym property is preserved under the direct sums $\ell_p (1 \leq p \leq \infty)$ and the fact that a Banach space with a bounded complete basis has the Radon-Nikodym property. Since X^* has the metric compact approximation property, by a well known result due to Johnson [15], $K(X^*, X^*)$ is an ideal in $L(X^*, X^*)$. Thus $K(X, X^{**})$ is an ideal in $L(X, X^{**})$ which establishes (i).

Since X has the Radon-Nikodym property but fails to have the metric compact approximation property, $K(X, X)$ is not an ideal in $L(X, X)$. Hence $K(X, X)$ is not an ideal in $L(X, X^{**})$. If $K(X, X)$ is not an ideal in $K(X, X^{**})$, then $K(X, X)$ is not an ideal in $L(X, X^{**})$ because $K(X, X^{**})$ is an ideal in $L(X, X^{**})$ which is impossible. Thus establishes (ii) and (iii). $\square$

Remark 2.1. From the above Theorem, the dual space X^* fails to have the metric compact approximation property with conjugate operators (although X^* has the metric approximation property). Lima [16] demonstrated that $K(Z, X)$ is an ideal in $K(Z, Y)$ for Banach spaces Z whenever X is an ideal in Y and X^* has the metric compact approximation property with conjugate operators. This addresses the question whether $K(Z, X)$ is an ideal in this $K(Z, X^{**})$ or not. It is clear since there is a norm one projection between the Banach space X and its dual, $K(Z, X)$ is an ideal in this $K(Z, X^{**})$ for all Banach space Z.

Theorem 2.2. *Let X be a closed subspace of a Banach space Y. The following statements are equivalent*

i. *X is an ideal in Y*

ii. $\overline{F(Z, X)}$ *is an ideal in* $\overline{F(Z, Y)}$ *for some Banach spaces Z.*

iii. $\overline{F(Z, X)}$ *is an ideal in* $\overline{F(Z, Y)}$ *for some Banach space $Z \neq \{0\}$.*
In particular $\overline{F(Z, X)}$ *is an ideal in* $\overline{F(Z, X^{**})}$ *for all Banach spaces X in Z.*

Proof. *$i \Rightarrow ii$.* Let X be an ideal of Y and $F(X, Y)$ be a class of finite rank compact operators. Since $Z^* = L(Z, \mathbb{F})$ is a vertical distribution, then there exists $\varepsilon, s > 0$ such that $\overline{F(Z, X)}$ and $\overline{F(Z, Y)}$ can be canonically identified with $Z^* \oplus_\varepsilon X$ and $Z^* \oplus_s Y$ respectively. Thus $\overline{F(Z, X)} \Delta \overline{F(Z, Y)}$ as required so that $(i) \implies (ii)$
$ii \Rightarrow iii$. since $\overline{F(Z, X)}$ is an ideal in $\overline{F(Z, Y)}$ for all Banach spaces Z then clearly $\overline{F(Z, X)}$ is an ideal in $\overline{F(Z, Y)}$ for some Banach space $Z \neq \{0\}$.
$iii \Rightarrow i$. Suppose $\overline{F(Z, X)}$ is an ideal in $\overline{F(Z, Y)}$. Let F be a finite dimensional subspace of Y. Let $z \in Z$ and $z^* \in Z^*$ be such that $\|z\| = \|z^*\| = z^*(z) = 1$. Denote $G = \{z^* \otimes y : y \in F\} \subseteq F(Z, Y)$. Let $\varepsilon > 0$ and let $V : G \to \overline{F(Z, X)}$ be an operator such that $\|V\| \leq 1 + \varepsilon$ and $V(S) = S$ for all $S \in G \cap \overline{F(Z, X)}$. Now define a map $U : F \to X$ by $Uy = (V(z^* \otimes y))\, z$. Then "Locally 1complements" X in Y. $\square$

Proposition 2.1. *Let X be a closed subspace of a Banach space Y and assume that $K(Z, X)$ is an ideal in $K(Z, X^{**})$ for some Banach space $Z \neq \{0\}$. Then X is an ideal in Y if and only if $K(Z, X)$ is an ideal in $K(Z, Y)$.*

Proof. The necessity condition is standard, so we only need the 'only if' part.

Let $\phi : X^* \to Y^*$ and $\Phi : K(Z,X)^* \to K(Z,X^{**})^*$ be Hahn-Banach extension operators. Define $\Psi : K(Z,X)^* \to K(Z,Y)^*$ by $(\Psi f)(T) = (\Phi f)(\phi^* \mid Y \circ T), f \in K(Z,X)^*, T \in K(Z,Y)$. Then Ψ is linear and $\|\Psi\| \leq 1$. Since $\phi^* x = x, x \in X$, we have $\phi^* \mid Y \circ T = T$ whenever $T \in (Z,X)$. Consequently, for any $T \in (Z,Y)$ and any $f \in K(Z,X)^*$, $(\Psi f)(T) = (\phi f)(T) = f(T)$ meaning that Ψf is an extension of f. Hence Ψ is a Hahn-Banach extension operator. $\square$

Remark 2.2. $K(Z,X)$ is not an ideal in $K(Z,X^{**})$ for all X and Z unless X is a norm one projection in its bidual. However $K(X,X^{**})$ is an ideal in $K(X,X^{***})$ because X^{**} is the range of norm one projection in X^{***}.

Theorem 2.3. *Let X be Banach space with the Radon-Nikodym property.Then the following statements are equivalent.*

a) X has the metric approximation property.

b) $F(X,X)[$ resp. $K(X,X)]$ is an ideal in the space $L(X,X)$.

c) $F(X,X)$ where $\alpha > 0$ and I is the identity operator (resp. $K(X,X)$) is an ideal in $\operatorname{Span}(F(X,X) \cup \{I_\alpha\})$ *(resp.* $\operatorname{Span} K(X,X) \cup \{I_\alpha\}$*).*

Theorem 2.4. *Let X be a Banach space .The following statements are equivalent:*

a) X has the metric approximation property (respectively the metric compact approximation property)

b) $F(Y,X)[$ resp. $K(Y,X)]$ is an ideal in the space $L(Y,X)$ for every Banach space Y.

c) $F(Y,X)$ [resp. $K(Y,X)$] is an ideal in the space $L(Y,X)$ for every separable Banach space Y.

d) $F(\hat{X},X)[$ resp. $K(\hat{X},X)]$ is an ideal in the space $L(\hat{X},X)$ for every equivalent renorming $\hat{X}$ of X.

Proof. a $\Rightarrow$ b is proved in [17]

b $\Rightarrow$ c and b $\Rightarrow$ d because $F(Y,X)$ [resp. $K(Y,X)$] is an ideal in the space $L(Y,X)$ for every Banach space Y implies that $F(Y,X)[$ resp. $K(Y,X)]$ is an ideal in the space $L(Y,X)$ for every separable Banach space Y and $F(\hat{X},X)$ [resp. $K(\hat{X},X)]$ is an ideal in the space $L(\hat{X},X)$ for every equivalent renorming $\hat{X}$ of X.

c $\Rightarrow$ a is proved as follows: Let $L \subseteq X$ be a separable subspace. It is well known that a separable ideal Y in X with $L \subseteq X$. Let $\varphi : Y^* \to X^*$. Let $\Psi : F(Y,X)^* \to K(Y,X)^*$ be Hahn-Banach extension operators and $i : F(Y,X) \to L(Y,X)$ be the inclusion map. Then $P = \Psi \circ i^*$ is an ideal projection. Let $I : Y \to X$ be the identity map. We find a net $(T_\alpha) \subseteq F(Y,X)$ with $\sup_\alpha \|T_\alpha\| \leq \|I\| = 1$ such that $x^*(T_\alpha y) \to (P(x^* \otimes y))(I) \forall y \in Y$ and $x^* \in X^*$, that is $T_\alpha \to I$ in the weak* topology. Let $\widehat{T_\alpha} = {T_\alpha}^{**} \circ \varphi^*|_x \in F(X,X)$ then $\left\|\widehat{T_\alpha}\right\| = \|T_\alpha\| \leq 1$ and $\widehat{T_\alpha}$ converges pointwise to the identity I_α on Y. It follows that X has a metric approximation property by definition.

d $\Rightarrow$ a. Let $Y = X$ and there exist $\Psi \in HB(F(X,X), L(X,X))$ such that $\Psi(x^* \otimes x) = x^* \otimes x$ for all $x^* \in X^*, x \in X$.Let $i : F(X,X) \to L(X,X)$ be an inclusive map and define $P = \Psi \circ i^*$. Using Theorem 5.4 in [18] with P as the ideal projection we conclude that X has the metric approximation property. $\square$

3 Ideals Through the Hahn-Banach Extension Operator

We provide an alternative approach to the definition of an ideal through the Hahn-Banach extension operator and discuss some ideal properties under this context. The next result is well known and can be found in [19].

Lemma 3.1. *Let F be a closed subspace of a Banach space E. The following statements are equivalent.*

(a) F is an ideal in E.

(b) *F is locally 1 -complemented in E, that is, for every finite dimensional subspace G of E and for all $\varepsilon > 0$, there is an operator $U : G \to F$ such that $\|U\| \leq 1 + \varepsilon$ and $Ux = x$ for all $x \in G \cap F$.*

(c) *There exists a Hahn-Banach Extension operator $\phi : F^* \to E^*$.*

By defining Hahn-Banach Extension operators on tensor product spaces the following result is immediate:

Theorem 3.2. *Let X be an ideal in Y and let Z be an ideal in W.Then $X \otimes Z$ is an ideal in $Y \otimes W$.*

Proof. Let $\phi : X^* \to Y^*$ and $\psi : Z^* \to W^*$ be Hahn-Banach Extension operators. Let $Q : Z^{***} \to Z^*$ be the canonical projection. Using the identifications $(X \otimes Z)^* = I(X, Z^*)$ and $(Y \otimes W)^* = I(Y, W^*)$ the map $\phi : I(X, Z^*) \to I(Y, W^*)$ defined by $\phi(T) = \psi \circ Q \circ T^{**} \circ \phi^* \backslash_Y$ is clearly a Hahn-Banach extension operator since $\phi^* x = x, x \in X$, and $\psi^* z = z$ for all $z \in Z$.

□

In the sequel we consider classes of ideals where additional constrains are imposed on the projections. Hereditary properties of these classes of ideals in relation to the basic properties of a general Banach space operator ideal are also discussed.

4 M-ideals

There is extensive literature concerning special class of ideals known as M-ideal [20, 21] and decompositions of Banach spaces by means of projections satisfying certain norm conditions. What are considered as special notions are contained in the following definitions.

Definition 4.1. Let X and Y be Banach spaces with $X \subseteq Y$. The annihilator of X is the set $X^{\perp} = \{y^* \in Y^* \mid y^*(x) = 0, \forall x \in X\}$

Definition 4.2. Let X be a real or complex Banach space

(a) A linear projection Ω is called an $\boldsymbol{M}$-projection if $\|x\| = \max\{\|\Omega x\|, \|x - \Omega x\|\}$ for all $x \in X$. It is equivalently an L-projection if $\|x\| = \|\Omega x\| + \|x - \Omega x\|$ for all $x \in X$.

(b) A closed subspace of $X \subset Y$ is called an $M-$ summand if it is the range of M-projection. It is equivalently an $L-$ summand if it is the range of $L-$ projection.

(c) A closed subspace of $X \subset Y$ is called an $\boldsymbol{M}$-ideal if $X^{\perp}$ is $L-$ summand in Y^*.

Remark 4.1. Every Banach space X contains the trivial M-summands $\{0\}$ and X. All the other M-summands are nontrivial. The same remark applies to L-summands and M-ideals.
There is an obvious duality between L-projection and M-projection: Ω is an L-projection on X if and only if Ω^* is an M-projection on X^*. Ω is an M-projection on X if and only if Ω^* is an L-projection on X^*

This remark yields the following characterization of M-projections which is useful in the sequel:
A projection $\Omega \in L(X)$ is an M-projection if and only if

$$\|\Omega x_1 + (Id - \Omega)x_2\| \leq \max\{\|x_1\|, \|x_2\|\} \text{ for all} x_1, x_2 \in X \tag{4.1}$$

In fact (4.1) means that the operator $(x_1, x_2) \mapsto \Omega x_1 + (Id - \Omega)x_2$ for $X \oplus_\infty X$ to X is contractive whence its adjoint $x^* \mapsto (\Omega^* x^*, (Id - \Omega)^* x^*)$ from X^* to $X^* \oplus_1 X^*$ is contractive, where (X $\oplus_p Y$ denotes the direct sum of two Banach spaces, equipped with l^p- norm). This means that Ω^* is an L-projection and Ω must be an M-projection. We note that there is only one M-projection of Ω with $X = \mathcal{R}(\Omega)(= \ker(Id - \Omega))$ if X is an M-summand and only one L-projection Ω with $X = \mathcal{R}(\Omega)(= \ker(Id - \Omega))$ if X is an L-summand. Consequently, there is a uniquely determined closed subspace of $\hat{X}$ such that

$$Y = X \oplus_\infty \hat{X}, \text{ respectively}$$
$$Y = X \oplus_1 \hat{X}$$

Then $\hat{X}$ is called the complementary $M-$ (Resp. $L-$)summand. The duality of $L-$ and M-projections may now be expressed as

$$Y = X \oplus_\infty \hat{X} \quad \text{if and only if } Y^* = X^\perp \oplus_1 \hat{X}^\perp$$
$$Y = X \oplus_1 \hat{X} \quad \text{if and only if } Y^* = X^\perp \oplus_\infty \hat{X}^\perp$$

It follows that M-summands are M-ideals and that the M-ideals X is an M-summand if and only if the L-summand complementary to $X^\perp$ is weak * closed. It is noted that the fact that X and $\hat{X}$ are complementary to L-summands in X means geometrically that B_X, the closed unit ball of X, is the convex hull of B_X and $B_{\hat{X}}$.

Proposition 4.1. *(a) If Ω is an M-projection on Y and Q is a contractive projection on Y satisfying $\mathcal{R}(\Omega) = \mathcal{R}(Q)$ then $\Omega = Q$.*

(b) If Ω is an L-projection on Y and Q is a contractive projection on Y satisfying $\operatorname{Ker}(\Omega) = \ker(Q)$ *then $\Omega = Q$.*

Proof. We first prove (b). Our argument follows that for an $L-$summand X in Y, there is a given $y \in Y$, one and only one best approximant x_0 in X, that $\|y - x_0\| = \inf_{x\in X} \|y - x\|$ namely the image of y under the L-projection on X. Let $X = \ker\,(\Omega)$. For $y \in Y$ we have $y - \Omega y \in \ker(\Omega) = \ker Q$, hence $\|y - (y - Qy)\| = \|Qy\|$

$$\begin{aligned} &= \|Q(y - (y - \Omega y)\| \\ &\le \|Q\|.\|\Omega y\| \\ &\le \|(y - (y - \Omega y)\| \end{aligned}$$

This means that $y - Qy \in \ker Q = \ker(\Omega)$ is at least as good an approximant to y in X as $y - \Omega y$ which is the best approximation. From the uniqueness of the best approximant one deduces $Qy = \Omega y$, thus $\Omega = Q$ as claimed. (a) follows from (b) since $\ker\left(\Omega^*\right) = \mathcal{R}(\Omega)^\perp = \ker\left(Q^*\right)$. ☐

The following result shows that M-ideals are "Hahn-Banach smooth"

Proposition 4.2. *Let X be an M-ideal in Y. Then every $y^* \in X^*$ has a unique norm preserving extension to a function $y^* \in Y^*$.*

Proof. By assumption, $X^\perp$ is an L-summand so that there is a decomposition

$$Y^* = X^\perp \bigoplus X^*.$$

But $X^\#$ can be explicitly decomposed since there are canonical isometric isomorphism $X^* \cong Y^*/X^\perp \cong X^\#$ so that $X^\# = \left\{y^* \in Y^*, \|y^*\| = \left\||y^*|_X : y^* \cdot x \neq 0, x \in X\right\|\right\}$ and the result follows. ☐

Remark 4.2. The above proposition enables us to consider a subspace J^* of X^* in a new decomposition given by

$$X^* = J^\perp \oplus_1 J^* \text{ where } J(X)\Delta X.$$

What follows in the sequel is an answer to the question whether an M-ideal in a Banach space induces an M-ideal in a subspace or a quotient space. We begin by availing the following general facts.

Lemma 4.1. *Let X and Z be closed subspaces in a Banach space Y.*

(a.) $X + Z$ is closed in Y if and only if $X^\perp + Z^\perp$ is closed in Y^* if and only if $X^\perp + Z^\perp$ is weak * Closed in Y^*. In this case $X^\perp + Z^\perp = (X \cap Z)^\perp$ and

$$(X + Z)/X \cong \mathrm{Z}/(X \cap Z), \left(X^\perp + Z^\perp\right)/Z^\perp \cong X^\perp/\left(X^\perp + Z^\perp\right).$$

(b.) Suppose $X^{\perp}$ is the range of a projection Ω such that $\left(\Omega Z^{\perp}\right) \subset Z^{\perp}$. Then the assertion of (a) hold. If Ω is contractive we even have

$$\left(X^{\perp}+Z^{\perp}\right)/X \cong Z/(X \cap Z)$$
$$\left(X^{\perp}+Z^{\perp}\right)/Z^{\perp} \cong X^{\perp}/\left(X^{\perp}+Z^{\perp}\right)$$

If $Id-\Omega$ is contractive we have $\quad (X+Z)/Z \cong X/(X\cap)$.

In the lemma that follows, isometry is established

Lemma 4.2. *Let Ω be a projection in Y and let $Z \subset X$ be a closed subspace. We suppose $\Omega(Z) \subset Z$ so that $\Omega\ Z : Z \to Z, y \mapsto \Omega y$ and $\Omega/Y : Y/Z \to Y/Z, y+Z \mapsto \Omega y + Z$ are well- defined projections. We have $\mathcal{R}(\Omega/Z) = \mathcal{R}(\Omega) \cap Z$ and $\mathcal{R}(\Omega/Z) = (\mathcal{R}(\Omega)+Z)/Z$. Also $\mathcal{R}(\Omega/Z) \cong \mathcal{R}(\Omega)/(\mathcal{R}(\Omega) \cap Z)$ if Ω is contractive. Moreover Ω/Z and Ω/Z are $L-(\text{resp}\, M-)$ projections if Ω is.*

Theorem 4.3. *Suppose that X^{**} or Y^{*} has the Radon-Nikodym property and that $K(X,Y)$ is an M-ideals in $L(X,Y)$*

(a) If X^{*} has the bounded compact approximation property with adjoint operators and Z is a closed subspace of Y, then $K(X,Z)$ is an M-ideal in $L(X,Z)$.

(b) If Y^{*} has the bounded compact approximation property with adjoint operators and E is a closed subspace of X, then $K(X/E,Y)$ is an M-ideal in $L(X/E,Y)$.

Using suitable Hahn-Banach Extension operators corresponding to ideal projections and using Feder-Sapher representation of the dual space of certain space of compact operators the following ideal properties of $K(X,Z)$ in $L(X,Z)$ for a closed subspace Z of Y are also evident.

Theorem 4.4. *[11] Let X and Y be Banach spaces such that X^{**} or Y^{*} has Radon-Nikodym property. If $K(X,Y)$ is an M-ideal in $L(X,Y)$, then for every closed subspace Z of $Y, K(X,Z)$ is an in M-ideal $L(X,Z)$.*

Since a reflexive Banach space has the Radon-Nikodym property, then if X is a reflexive Banach space and Z is a closed subspace of a Banach space Y. If $K(X,Y)$ is an M-ideal then $K(X,Z)$ is an M-ideal in $L(X,Z)$.

Theorem 4.5. *[11]: Let X and Y be Banach spaces. Suppose that $K(X,Y)$ is an M-ideal in $L(X,Y)$*

(a) If X^{*} has metric compact approximation with adjoint operators, then $K(X,F)$ is an M-ideal in $L(X,F)$ for a closed subspace F of Y.

(b) If Y^{*} has the compact approximation property then $K(X/E,Y)$ is an M-ideal in $L(X/E,Y)$ for every closed subspace E of X.

In [14] and [19], it is shown that a closed subspace X of a Banach space Y is an M-ideal in Y if and only if for every finite dimensional subspace G in Y and every $\epsilon > 0$, there exists a linear operator $U : G \to X$ such that $Ux = x$ for all $x \in G \cap X$ and

$$\|Ux + y - Uy\| \leq (1+\epsilon)\max(\|x\|, \|y\|) \text{ for all } x, y \in G.$$

We now give a generalization of notions of $M-$ ideals in the tensor product of Banach algebras and their extensions to quotient images. These characteristics lead us to the classes of ideals in the next section.

5 u-Ideals and h-Ideals

Suppose Y is a real or complex Banach space, a closed subspace X of Y is a summand if there exists a contractive projection of Y onto X. We further say that a closed subspace X of Y is u-summand if there is a subspace Z so that $X \oplus Z = Y$, and if $x \in X, z \in Z$ then $\|x+z\| = \|x-z\|$.

If Y is a complex Banach space we say that X is a h-summand with h-complement Z if $X \otimes Y = Z$ and if $x \in X, z \in Z$ and $|\lambda| = 1$ then $\|x + \lambda Z\| = \|x + z\|$. If X is u-summand then the induced projection $P : Y \to X$ with $P(Y) = X$ and ker $P = Z$ satisfies $\|I - 2P\| = 1$. Likewise if X is an h-summand then $\|I - (1+\alpha)P\| = 1$ whenever $|\alpha| = 1$ which is equivalent to saying that P is Hermitian.
Next, we provide a lemma in which we show that the projection P defined above is unique.

Lemma 5.1. *Suppose X is a closed subspace of Y. Then the projection P of Y onto X satisfying $\|I-2P\| = 1$ is unique.*

Proof. Suppose that we have two projections P and Q such that $\|I - 2P\| = \|I - 2Q\| = 1$, then

$$\begin{aligned}(I-2P)(I-2Q) =&(I-2Q) - 2P(I-2Q)\\ &= I - 2Q - 2P + 4PQ\end{aligned}$$

But $Q(Y) = X$, we have $(PQ)Y = P(QY) = Qy$ where $y \in Y$ and $Qy \in X$, therefore

$$(I-2P)(I-2Q) = I - 2Q - 2P + 4Q$$

Thus we have

$$\begin{aligned}((I-2P)(I-2Q))^2 &= (I + 2(Q-P)(I + 2(Q-P)))\\ &= I + 2(Q-P) + 2((Q-P)(I + 2(Q-P)))\\ &= I + 2(Q-P) + 2(Q-P) + 4(Q-P)^2\\ &= [I + 2(Q-P)]^2\\ &= I + 2.2(Q-P)\\ ((I-2P)(I-2Q))^3 &= (I + 4(Q-P))(I + 2(Q-P))\\ &= I + 2(Q-P) + 4(Q-P) + 8(Q-P)^2\\ &= I + 2.3(Q-P) \text{ etc.}\end{aligned}$$

In general, we have $((I-2P)(I-2Q))^n = I + 2 \cdot n(Q-P)$ and since

$$\|I - 2n(Q-P)\| = \|I - 2n(P-Q)\| \geq \|I - 2n\|\|P-Q\| \to \infty \text{ as } n \to \infty \text{ if } \|P-Q\| \neq 0.$$

And $\|((I-2P)(I-2Q))^n\| \leq \|I-2P\|^n\|I-2Q\| = 1$. We have a contradiction unless $P = Q$.

□

Assuming that X is a u-ideal in Y. Let V be u-complement of $X^{\perp}$ in Y^*. Let P be a projection on Y^*, so that $\|I - 2P\| = 1$ and where the range $\mathcal{R}(\mathcal{P}) = \mathcal{V}$ and ker $P = X^{\perp}$, then we have the following Lemma:

Lemma 5.2. *If X is a u-ideal in Y then X is a u-summand if and only if V is weak*-closed.*

Proof. If V is weak *-closed then P is weak *-continous and so $P = Q^*$ where $\|I - 2Q\| = 1$ and $Q(Y) = X$. Conversely suppose X is u-summand and let Q be a projection onto X with $\|I - 2Q\| = 1$. Then $I - Q^*$ has range $X^{\perp}$ and so $I - Q^* = I - P$ has P is weak *-continuous. □

Motivated by the above lemma we say that X is a strict u-ideal or as in the complex space a strict h-ideal if V is a norming subspace of Y^*. Refer to [19] for much literature in this direction.

Definition 5.1. If X is an arbitrary Banach space and $x^{**} \in X^{**}$. We define its u-constant $k_u(x^{**})$ to be the infimum of all such $a's$ such that we can write $x^{**} = \sum_{n=1}^{\infty} x_n$ in the weak *-topology with $x_n \in X$ and such that for any n and $\theta_k = \pm 1$ for $1 \leq k \leq n$ we have $\left\|\sum_{n=1}^{n} \theta_k x_k\right\| \leq a$. We set $k_u(x^{**}) = \infty$ if no such a exists.

Definition 5.2. Let $Ba(X)$ be the collection of $x^{**} \in X$ such that there is a sequence (x_n) in X with $\lim x_n = x^{**}$. Then X has property (u) if $x^{**} \in Ba(X)$ has $k_u(x^{**}) \leq \infty$. Then by the closed graph theory, there exists some c so that $k_u(x^{**}) \leq c\|x^{**}\|$ for all $x^{**} \in Ba(X)$. We denote the least of such constant by $k_u(X)$.

Definition 5.3. If X is a complex Banach space, we define $k_h(x^{**})$ to be the infimum of all $a's$ such that $x^{**} = \sum_{n=1}^{\infty} x_n$ and for any n, and any $|\theta_k| = 1$ for $1 \leq k \leq n$, we have $\left\|\sum_{n=1}^{n} \theta_k x_k\right\| \leq a$. clearly $k_u(x^{**}) \leq k_h(x^{**}) \leq 2k_u(x^{**})$.

Definition 5.4. If X has property (u) we define $k_h(X)$ to be the least constant c so that $k_h(x^{**}) \leq c\,\|x^{**}\|$ for $x^* \in Ba(X)$. Thus $k_u(X) \leq k_h(X) \leq 2k_u(X)$.

We are now in a position to show the lemma that follows:

Lemma 5.3. *Suppose that X is a u-ideal (respectively an h-ideal) in Y. Suppose that $y \in Y$ and $\epsilon > 0$. Let A be a convex subset of X such that Ty is in the weak *-closure of A and that B is any compact subset of X. Then there exists $x \in A$ such that $\|y - (1+\lambda)x - \lambda Z\| < \|y+z\| + \varepsilon$ whenever $-1 \leq \lambda \leq 1$ (respectively $|\lambda| \leq 1$) and $Z \in B$.*

Proof. We may assume that $0 \in B$. Let $M = \max\{\|x\| : Z \in B\}$ and pick $0 < \delta < 1$ so that $(M+4+2\|y\|)\delta < \varepsilon$. Let $\{\lambda_1, \ldots, \ldots, \lambda_m\}$ be a δ-net for the closed unit disk, which we suppose includes zero and let $\{z_1, \ldots, ..z_n\}$ be a net for B, also including zero. For any subset J of $\Omega = [m] \cap [n]$, define H_J to be the set of $x \in A$ such that

$$H_J = \|y - (1+\lambda_j)\,x - \lambda_j Z_k\| < \|y + Z_k\| + \delta$$

whenever $(j,k) \in J$. Thus $H_\theta = A$. We proceed to show that H_Ω is none empty. Pick any $(j,k) = \Phi$ where $K = J \cup \{(i,j)\}$. However $A^1 = W \cap H_J$ is convex and Ty is in its weak *-closure. Thus $T(y + Z_k)$ is in the weak *-closure of $A^1 + Z_k$. There exists $x \in W \cap H_J$ such that $\|y + Z_k - (1+\lambda_j)(x + Z_k)\| < \|y + Z_k\| + \delta$. On reorganizing this implies that $x \in W \cap H_K$ which is contrary to the assumption that $(j,k) = \Phi$. It then follows that $H_\Omega \neq \Phi$.

Next pick any $x \in H_\Omega$ then if $x \in B$ and $|\lambda| \leq Y$, we may find $(i,j) \in \Omega$ such that $|\lambda - \lambda_j| \leq \delta$ and $|Z - Z_j| \leq \delta$. Thus

$$\begin{aligned}
\|\mathrm{y} - (1+\lambda)x - \lambda Z\| &\leq \|y - (1+\lambda_j)\,x - \lambda_j Z_k\| + \delta\,(1 + \|x\| + \|Z_k\|) \\
&\leq \|y + Z_k\| + \delta\,(1 + \|x\| + \|Z_k\|) \\
&\leq \|y + Z\| + \delta\,(3 + \|x\| + \|Z_k\|)
\end{aligned}$$

Zero appears in both δ-net so we have $\|y - x\| \leq \|y\| + \delta$ thus $\|x\| \leq 2\|y\| + 1$. hence

$$\begin{gathered}
\delta\,(3 + \|x\| + \|Z_k\|) \leq (M + 4 + 2\|y\|)\delta < \varepsilon \\
\delta\,(3 + \|x\| + \|Z_k\|) < \varepsilon
\end{gathered}$$

□

In the following proposition, we characterize u-ideals and h-ideals.

Proposition 5.1. *Let Y be a Banach space and let X be a closed subspace of Y. For X be a $u-$ ideal (respectively h-ideal) in Y, it is necessary and sufficient that for every finite-dimensional subspace F of Y and every $\varepsilon > 0$, there is a linear map $L : F \to X$ such that $Lx = x$ for $x \in F \cap X$ and $\|f - 2Lf\| \leq (1+\varepsilon)\|f\|$ for every $f \in F$ (respectively $\|f - (1+\lambda)Lf\| \leq (1+\varepsilon)\|f\|$ for every $f \in F$ and for every λ such that*

$$|\lambda| = 1).$$

Proof. We prove the result in the u-ideal case since for the case of h-ideal the result holds with slight modifications.

Suppose that X is a u-ideal in Y and that F is a finite dimensional subspace of Y. Then we claim that $L(F,X)$ is a u-ideal in $L(F,Y)$. In fact $L(F,X)^*$ can be identified with $F \otimes_\Pi Y^*$ and so we can induce a projection P on it by $P(f \otimes_\Pi y^*) = f \otimes Py^*$. It is clear that $\|1 - 2P\| = 1$ and that $\ker P = L(F,X)^\perp$. Further P induces a map $T : L(F,Y) \to L(F,X^{**})$ in the usual way so that for any $\Phi \in L(F,X)^*$ we have $T(L)(\Phi) = P(\Phi)(L)$.Let

$J : F \to Y$ be the identity map and $\mathcal{A}$ be the collection of all $\mathcal{L} \in L(F, X)$ such that $\mathcal{L}x = x$ for all $x \in P \cap X$. Suppose $T(J)$ is not in the weak *-closure of $\mathcal{A}$, there exists $\Phi \in P \cap X$. Suppose $T(J)$ is not in the weak *-closure of $\mathcal{A}$, then there exists $\Phi \in L(F, X)^*$ so that $\sup\limits_{L \in \mathcal{A}} \mathcal{R}\Phi(L) = \alpha < \mathcal{R}T(J)(\Phi)$. Clearly $\Phi(S) = 0$ if $S = 0$ on $F \cap X$. It follows that we can write $\Phi = \sum_{j=1}^{m} f_j \otimes {x_j}^*$ where $\{f_j\}$ is abasis for $F \cap X$, and ${x_j}^* \in X^*$. Let ${y_j}^*$ be extensions of ${x_j}^*$ to Y^*, then

$$\langle \Phi, T(J) \rangle = \left\langle J, P\left(\sum_{j=1}^{m} f_j \otimes y_j^*\right)\right\rangle = \sum_{j=1}^{m} \langle J, f_j \otimes P{y_j}^* \rangle = \sum_{j=1}^{m} \langle f_j, P{y_j}^* \rangle = \sum_{j=1}^{m} {x_j}^* (f_j).$$

Now letting S be any projection of F onto $F \cap X$ we find that $\Phi(S) = T(J)(\Phi)$. Thus $T(J)$ is in the weak *-closure of $\mathcal{A}$ and so there exists $\mathcal{L} \in \mathcal{A}$ so that

$$\|J - 2\mathcal{L}\| < 1 + \varepsilon$$

Conversely, suppose for every finite-dimensional F and $\varepsilon > 0$, there exists $\mathcal{L} = \mathcal{L}_{F,\varepsilon} : F \to X$ so that $\mathcal{L}x = x$ for $x \in F \cap X$ and $\|f - 2\mathcal{L}f\| \leq (1 + \varepsilon)\|f\|$ for $f \in F$. We regard the collection of all pairs (F, ε) as a directed set in the obvious way. Extending $L_{F,\varepsilon}$ to a nonlinear operator $L_{F,\varepsilon} : Y \to X$ by setting $L_{F,\varepsilon}(x) = 0$ for $x \notin F$. By compactness argument, we find a subnet (L_d) of $L_{F,\varepsilon}$ so that for every $y^* \in Y^*, y \in Y, \lim_j y^* (L_d y) = h(y^*, y)$ exists. Then $y \to h(y^*, y)$ is linear and bounded and so we can define $Py^* \in Y^*$ by $\langle y, Py^* \rangle = h(y^* y)$. Further P is linear, $\ker P = X^\perp$ and $\|I - 2P\| = 1$, the ideal property. □

6 Embeddments in the Biduals

We consider the u and $h-$ ideals of X embedded in the biduals X^{**} where X is a Banach Algebra.

Definition 6.1. We will say that X is a u-ideal (respectively an h-ideal) if X is a u-ideal (respectively an h-ideal) in X^{**} for the canonical embedding.

Proposition 6.1. *A Banach space X is a u-ideal (respectively an h-ideal) if given a finite-dimensional subspace F of X^{**} and $\varepsilon > 0$ there exists a linear map $L : F \to X$ so that $Lx = x$ for $x \in X \cap F$ and $\|f - 2Lf\| \leq (1 + \varepsilon)\|f\|$ for $f \in F$ (respectively $\|f - (1 + \lambda)Lf\| \leq (1 + \varepsilon)\|f\|$ for $|\lambda| = 1$ and $f \in F$).*

Remark 6.1. Suppose X is a Banach space so that for every $\varepsilon > 0$ there is a u-ideal (respectively an h-ideal) Y so that X is $(1 + \varepsilon)$-isomorphic to a $(1 + \varepsilon)$-complemented subspace of Y. Then X is a u-ideal (respectively an h-ideal).

Next, we turn to general Theory of h-ideals. We choose not to put any restrictions. We shall exploit the Hermitian operators on the general Banach space. The following general result will be the first in this direction.

Lemma 6.1. *Let X be an arbitrary complex Banach space. Let $H : X^{**} \to X^{**}$ be a Hermitian operator such that $Hx = 0$ for $x \in X$ with $\|x^{**}\| = 1$, then there exists a sequence (x_d) in X such that $\lim_d x_d = Tx^{**}$ weak * and for every $|\lambda| = 1$, $\limsup_d \|x^{**} - (1 + \lambda)x_d\| \leq 1$.*

Proof. We assume first that X is an h - ideal. Next we define for all real t, the operator $\exp(itP)$ invertible and isometric on X^{**} which satisfies $\exp(itP)x = x$ for all $x \in X$.
We need to show that there exists a projection T onto $B_a(X)$. Now $Tx^{**} = x^{**}$ for $x^* \in B_a(X)$, we claim that if $x^{**} \in B_a(X)$ is such that the ker x^{**} is norming then $x^{**} = 0$. This follows from the fact that $K_h(x^{**}) = \|x^{**}\|$. Now if X is separable there is a separable norming subspace M in X^*. If $x^{**} \in S_{X^{**}}$ and $\varepsilon > 0$ we can find $\chi \in B_a(X)$ with $K_h(\chi) < 1 + \varepsilon$ and such that $\chi(f) = Tx^{**}(f)$ for $f \in M$ and $\chi_1(x^*) = Tx^{**}(x^*)$. This follows by applying the same argument to the span of M and x^*. Thus $\chi_1 - \chi \in B_a(X)$ and $M \subset \mathrm{Ker}(\chi_1 - \chi)$. It also follows that $\chi_1 = \chi$ and hence that $T\chi^{**}(x^*) = \chi(x^*)$ for all $x^* \in X^*$. Thus $Tx^{**} = \chi \in B_a(X)$. Hence T is a Hermitian projection onto $B_a(X)$.

Conversely, We define $P : X^{***} \to X^{***}$ by $P = T^{*}\pi$. It is clear that πT^{*} is a projection of X^{***} onto X^{*} and so P is a projection whose kernel is $X^{\perp}$. So, for each $x^{**} \in S_{X^{*}}$ there exists a net (x_d) with $\lim_{im} x_d = Tx^{**}$ weak * and so $|\lambda| = 1$ then $\limsup \|x^{**} - (1+\lambda)x_d\| \leq 1$. Thus, it follows that $\|1 - (1+\lambda)P\| = 1$ if $|\lambda| = 1$ and this shows that $\|\exp(itP)\| = 1$ for all real t, that is, P is Hermitian. □

The Theorem that follows characterizes h-ideals. First, we provide the Lemmatta that shall be used in its proof.

Lemma 6.2. *T is Hermitian on X^{**} and $T(x) = x$ for $x \in X$*

Lemma 6.3. *Let X be arbitrary real or complex Banach space and suppose $x^{**} \in S_{X^{**}}$ satisfies $k_u(x^{**}) < 2$. Then ker x^{**} cannot be a norming subspace of X^{*}.*

Proof. Let $x^{**} = \sum x_n$ weak * where if $\theta_k = \mp 1$ then, $\left\|\sum_{k=1}^{n} \theta_k x_k\right\| \leq 2 - \delta$ for some $\delta > 0$. If $S_n = \sum_{k=1}^{n} x_k$ then $\|x^{**} - 2S_n\| \leq 2 - \delta$. Then there is a sequence of convex combinations t_n converges weak * to x^{**} and $\lim \|t_n\| = 1$. Thus $\|x^{**} - 2t_n\| \leq 2 - \delta$, which leads to the fact that if $x^{*} \in \ker x^{**}$ and $\|x^{*}\| = 1$ then $|x^{*}(t_n)| \leq 1 - \delta/2$. Thus $kerx^{**}$ cannot be norming.

□

Theorem 6.4. *Let X be a h-ideal. Then the following are equivalent*

1. *X is a strict h-ideal*
2. *X^{*} is an h-ideal*
3. *Every separable subspace of X has a separable dual.*
4. *$\|I - \lambda\pi\| \leq 1$ if $|\lambda - 1| \leq 1$*
5. *X contains no copy of l_1*

Next, we consider how we can identify h-ideals and this we do by considering first subspace.

Theorem 6.5. *Let X be a separable h - ideal and T be the induced Hermitian projection of X^{**} onto $Ba(X)$. If Z is a subspace of X such that $Z^{\perp\perp}$ is T-invariant then Z is an h-ideal.*

Proof. $Z^{\perp\perp}$ can be identified with Z^{**} and T restricted to an Hermitian projection on Z^{**} whose range include Z. If $z^{**} \in Z^{**}$ with $\|z^{**}\| = 1$ then Tz^{**} is in the weak *-closure of B_z and so there is a net (z_d) in Z with $\lim z_d = TZ^{**}$ week* and $\limsup_d \|Z^{**} - (1+\lambda)z_d\| \leq 1$ whenever $|\lambda| = 1$ and the result follows. □

Definition 6.2. We say that a separable $h-$ ideal is non degenerate if whenever $\chi \in \ker T$ and $x^{**} \in Ba(X)$, then $\|\chi + x^{**}\| = \|\chi\|$ implies $x^{**} = 0$.

Theorem 6.6. *Let X be separable non degenerate h - ideal.Then a closed subspace Z of X is an h-ideal If and only if $Z^{\perp\perp}$ is T-invariant.*

Proof. Consider a h - ideal Z and $T_z : Z^{**} \to Ba(X)$ be the associated Hermitian projection. The l_1-sum $X \oplus_1 Z$ is also an h - ideal and the associated projection of $X^{**} \oplus_1 Z^{**}$ onto $Ba(X) \oplus_1 Ba(Z)$ is given by $T \oplus T_Z$. Suppose $\chi \in Z^{**}$ satifies $T_z\chi = 0$ and $\|\chi\| = 1$. Identifying Z^{**} with $Z^{\perp\perp} \subset X^{**}$ in the natural way to consider χ in X^{**}. Then $T\chi \in B_a(X)$ and so there is a sequence (u_n) in X converging weakly to $T\chi$. Let $\xi = \chi - T\chi$, let $C_n \subset X \oplus_1 Z$ be the set of all (x, z) such that $z - x \in C_n\{u_k : k \geq n\}$ then (ξ, χ) is the weak * - closure by each C_n. If $\delta > 0$ then (ξ, χ) is also in weak *-closure of $A_n = \{(x, z) \in C_n : \|x\| \leq (1+\ \delta)\|\xi\|, \|Z\| \leq 1 + \delta\}$. In fact if $B = \{(x, z) : \|x\| \leq \|\xi\|, \|Z\| \leq 1\}$, then for any weak *-neighbourhood W of $(\xi, \chi), 0$ is in the weak-closure of $(W \cap C_n) - B$ and hence also in norm - closure. Hence $0 \in (W \cap C_n) - (1+\delta)B$, whence $W \cap C_n \cap (1+\delta)B$ is nonempty. It follows that we can pick $(x_n, z_n) \in A_n$ so that for all scalars $\alpha_1, \ldots, \ldots, \alpha_n$ and all $n \in \mathbb{N}$,

$$\left\|\sum_{k=1}^{n} \alpha_k x_k\right\| + \left\|\sum_{k=1}^{n} \alpha_k z_k\right\| \geq (1-\delta)(1+\|\xi\|)\sum_{k=1}^{n} |\alpha_k|\,.$$

By construction $\lim\left(z_n - x_n\right) = T\chi$ weak * and so $\lim\left(z_{2n} - z_{2n+1} - x_{2n} + x_{2n+1}\right) = 0$ weakly. Thus, there exists $n \in \mathbb{N}, \beta_k \geq 0$ such that $\sum \beta_k = 1$ and

$$\left\|\sum_{k=1}^{n} \beta_k\left(z_{2k} - z_{2k+1}\right) - \sum_{k=1}^{n} \beta_k\left(x_{2k} - x_{2k+1}\right)\right\| \leq \delta.$$

It follows that

$$\left\|\sum_{k=1}^{n} \beta_k\left(z_{2k} - z_{2k+1}\right)\right\| \leq 2(1+\delta)\|\xi\| + \delta$$

and hence that that

$$\left\|\sum_{k=1}^{n} \beta_k\left(z_{2k} - z_{2k+1}\right)\right\| + \left\|\sum_{k=1}^{n} \beta_k\left(x_{2k} - x_{2k+1}\right)\right\| \leq 4(1+\delta)\|\xi\| + \delta$$

Hence

$$2(1-\delta)(1+\|\xi\|) \leq 4(1+\delta)\|\xi\| + \delta$$

and as $\delta > 0$ is arbitrary this implies as $\delta \to 0$ we have $2(1+\|\xi\|) \leq 4\|\xi\|$, we have $1 \leq \|\xi\|$. Hence $\|\xi\| = 1$ and $\xi + T\chi = T\chi$ and X is non degenerate $h - ideal$ and this in turn implies $T\chi = 0$. It follows immediately that $T_z\chi = \mathrm{T}\chi$ for any $\chi \in Z^{\perp\perp}$ and so $Z^{\perp\perp}$ is T-invariant. □

7 u-ideals

For the $u-$ideals, we first consider the case when X contains no subspace isomorphic to l_1.

Proposition 7.1. *Let X be a Banach space containing no copy of l_1 which is a u-ideal. Then V is weak *-dense in X^{***}.*

Proposition 7.2. *Let X be a Banach space containing no copy of l_1. Suppose P is a projection on X^{***} such that* $\ker P = X^{\perp}$ *and $\|P\| = 1$. Let $V = P\left(X^{***}\right)$. Then $V \cap X^*$ is norming for X.*

Proof. We consider the associated map T. It is clear from the definition that if $x^* \in X^*$ and $T^*x^* = x^*$ then $x^* \in V$. Now for each $\chi \in X^{**}$ consider the set

$$E_\chi = \left\{x^* \in X^* : T\chi\left(x^*\right) = \chi\left(x^*\right)\right\}.$$

Now χ is of the first Baire class on $\left(B_{X^*}, w^*\right)$ and therefore the set $C_*(\chi)$ of points of continuity is a dense G_d-set. Assume $x^* \in C_*(\chi)$. Let $v = Px^*$. Then there is a net $\left(x_a^*\right)$ in B_{X^*} converging in the weak * topology of X^{***} to v. However, $\left(v - x^*\right) \in X^{\perp}$ so that ${x^*}_d$ converges for $\sigma\left(X^*, X\right)$ to x^*. Thus $v(\chi) = \lim_d \chi\left(x_d^*\right) = \chi\left(x_d^*\right)$ so that $Px^*(\chi) = \chi\left(x^*\right)$. Now $T\chi\left(x^*\right) = Px^*(\chi)$ so, we conclude that $x^* \in E_{\chi^*}$. Hence E_χ is norming and further this implies that $H = \cap_{X \in X} \ldots . E_X$ is also norming provided that $H \subset V \cap X^*$. □

In the Theorem that follows we look at a more general case of separable $u -$ ideal and prove the result

Theorem 7.1. *Let X be a separable u-ideal such that $k_u(X) \leq 2$. Then $k_u(X) = 1$ and $Ba(X)$ is a u-summand in X^{**}.*

Proof. If $x^{***} \in Ba(X)$ is such that $\ker x^{***}$ is norming then $x^{***} = 0$. Suppose $x^{**} \in X^{**}$. Let M be a separable norming subspace of X^*. Suppose $\varepsilon > 0$, then there exists $\chi \in X^{**}$ with $k_u(\chi) \leq \|x^{**}\| + \varepsilon$ so that $\chi(f) = Tx^{**}(f)$ for all $f \in M$. This leads to the conclusion that $\chi = Tx^{**}$. Thus T maps X^{**} into $B_a(X)$. Next if $x^{**} \in B_a(X)$, then the set of $x^* \in B_{X^*}$ such that $x^{**}\left(x^*\right) = Px^{**}\left(x^*\right)$ contains a weak $^*-$ dense G_d-subset. Hence $x^{**} - Tx^{**}$ vanishes on a norming subspace of X^* and is in $B_a(X)$. Hence $Tx^{**} = x^{**}$. Thus T is a projection of X^{**} onto $B_a(X)$ and of course $\|I = 2T\| = 1$. It further follows that if $x^{**} \in B_a(X)$ then $k_u\left(x^{**}\right) = \|x^{**}\|$. □

Now, let X be a closed subspace of a Banach space Y and let i_X be the natural embedding $i_X : X \to Y$. If P is a norm one projection on Y^* with $\ker P = X^\perp$ we may define a norm one operator $T : Y \to X^{**}$ by letting $\langle {i_X}^* y^*, T(y)\rangle = \langle y, P(y^*)\rangle$ for all $y \in Y$ and $y^* \in Y^*$. Then $T(x) = x$ for all $x \in X$ and if $(I - 2P)$ is an isometry then $\|y - 2i_{x^{**}} T(y)\| = \|y\|$ for all $y \in Y$. Further more if we let $V = P(Y^*)$, then X being a u-ideal in Y means that $Y^* = V \oplus X^\perp$ and $\|v + \eta\| = \|v - \eta\|$ for all $v \in V$ and $\eta \in X^\perp$.
consequently, we have the following results:

Lemma 7.2. *Let X be a closed subspace of a Banach space Y. If X is a u-ideal in Y then for every $\varepsilon > 0$, $y \in Y$ and $x \in X$ there is $x_0 \in X$ such that $\|y + x - 2x_0\| = \|y - x\| + \varepsilon$.*

Theorem 7.3. *Let X be a closed subspace of a Banach space Y. The following statements are equivalent.*

(a) X is a u-ideal in Y.

(b) X is a u-ideal in Z for every subspace Z of Y with $\dim Z/X < \infty$.

(c) X is a u-ideal in Z for every subspace Z of Y with $\dim Z/X \leq 2$.

8 Co-Dimension One

Let X be a closed subspace of a Banach spaceY. For $y \in Y \backslash X$ we can define the set of best approximants $P_y = \left\{x^{**} \in X^{**} : \|y - {i_X}^{**}(x^{**})\| = d\left(y, X^{\perp\perp}\right)\right\}$ .P_y is a non-empty weak $^*-$ compact convex subset of X^{**}. We give a number of lemmas crucial in one of our main results.

Lemma 8.1. *Let X be a closed subspace of a Banach space Y.Then, $I - P$ has norm one if and only if $T(y) \in P_y$ for all $y \in Y$.*

Proof. If $T(y) \in P_y$ then,

$$\begin{aligned}\|I-P\| &= \sup_{y^* \in B_{Y^*} \ldots}\ \sup_{y \in B_Y} |\langle y, y^* - Py^*\rangle| = \sup_{y^* \in B_{Y^*}}\ \sup_{y \in B_Y} |\langle y^*, y - {i_X}^{**}Ty\rangle| \\ &\leq \sup_{y \in B_Y} \|{i_X}^{**}Ty - y\| \leq \sup_{y \in B_Y} d\left(y, X^{\perp\perp}\right) \leq \sup_{y \in B_Y} \|y - 0\| \leq 1.\end{aligned}$$

So we have $\|I - P\| \leq 1$.
Conversely if $\|I - P\| = 1$ then,

$$\|y - {i_x}^{**}Ty\| = \|(y - x) - {i_x}^{**}T(y - x)\| \leq \|y - x\|$$

So that $T(y) \in P_y$. ☐

Remark 8.1. If $\|I - P\| = 1$ then both $I - P$ and P have norm one. $T(y) \in P_y$ is the Center of symmetry.

Definition 8.1. An element C in a convex set J is a center of symmetry if $2C - x \in J$ for all $x \in J$. C is a centre of symmetry if and only if $K - C$ is symmetric about the origin. This center of symmetry is unique.

Lemma 8.2. *Let X be a closed subspace of a Banach space Y. If X is a u-ideal in Y then $T(y)$ is a center of symmetry in P_y for all $y \in Y$.*

Proof. Let $x^{**} \in P_y$. Since $I^* - 2P^*$ is an isometry we have $d\left(y, X^{\perp\perp}\right) = \|y - {i_X}^{**}(x^{**})\|$

$$\begin{aligned}&= \|y - i_x^{**}(x^{**}) - 2P^*(y - i_x^{**}(x^{**})\| \\ &= \|y + i_x^{**}(x^{**} - 2T(y))\|\end{aligned}$$

So that $2T(y) - x^{**} \in P_y$. ☐

The next result emphasizes the local properties of ideals. They are aimed at depicting the global properties of the said ideals.

Proposition 8.1. *Let X be a closed subspace of a Banach space Z such that $Z/X < \infty$, then X is a u-ideal in Z if and only if for every subspace $W \subseteq X$ of finite co-dimension X/W is a u-ideal in Z/W.*

Proof. Let X be a u-ideal in Z and let $W \subseteq X$ be a finite co-dimensional subspace. Let $T : Z \to X^{**}$ with $T(x) = x$ for all $x \in X$ and $\|z - 2T(z)\| = \|z\|$ for all $z \in Z$. Let $Q_W : Z \to Z/W$ be the quotient map. Define $T_W : Z/W \to X/W = (X/W)^{**}$ by

$$T_W\left(Q_W(Z)\right) = Q_W^{**}\left(i_X^{**}(T(z))\right)$$

Which is well defined since $T_W(0) = {Q_W}^{**}\left({i_X}^{**}(T(W))\right) = Q_W(W) = 0$. We have

$$\sup_{W_W(z)\in B_{z/W}} \|Q_W(z) - 2T_W\left(Q_W(z)\right)\| = \sup_{z\in B_z} \|{Q_W}^{**}(z) - 2{Q_W}^{**}\left({i_X}^{**}(T(z))\right)\| \leq 1$$

And for $Q_W(x) \in X/W$

$$T_W\left(Q_W(x)\right) = {Q_W}^{**}\left({i_X}^{**}(T(x))\right) = {Q_W}^{**}\left({i_X}^{**}(x)\right) = Q_W(x)$$

By finite dimensionality of X/W and weak *-continuity of both ${Q_W}^{**}$ and ${i_X}^{**}$, we get that T_W is contained in X/W. Thus, X/W is a u-ideal in Z/W.

Conversely, let C_X denote the set of all finite-co-dimensional subspaces in X and suppose X/W is a u-ideal in Z/W for all $W \in C_X$. Let $W \in C_X$ and $Q_W : Z \to Z/W$. We have $\dim X/W < \infty$ and $\dim Z/W < \infty$ and as above we consider X/W as a subspace of Z/W and identify $Q_W(X)$ with X/W. We can therefore identify $(X/W)^{\perp}$ with $X^{\perp} \subseteq (Z/W)^* = (W)^{\perp}$ in Z^*. By assumption there is a projection $P_W : W^{\perp} \to W^{\perp}$ with $\operatorname{Ker} P_w = X^{\perp}$. Let $\mathcal{U}$ be an ultrafilter refining, the reverse order filter on C_X. Define $P : Z^* \to Z^*$ by

$$P\left(z^*\right) = \omega^* - \lim_u P_W\left(z^*\right).$$

Then

$$\|z^* - 2Pz^*\| \leq \lim_u \|z^* - 2P_W z^*\| \leq \|z^*\|$$

and $\ker P = X^{\perp}$ since $z^* \in Z^* \backslash X^{\perp}$ is in $W^{\perp} \backslash X^{\perp}$ eventually.

□

The lemma that follows shows that the ball intersection property is inherited by quotients.

Lemma 8.3. *Let X be a closed subspace of a Banach space Y and let $y \in Y \backslash X$. Assume that $X \cap_{i=1}^{3} B_Y\left(y + x_i, \|y - x_i\| + \varepsilon\right) \neq \emptyset, \varepsilon > 0$ for every collection of three points $(x_i)_{i=1}^{3} \subset X$.If W is a finite co-dimensional subspace of X then X/W has the property*

$$X \cap_{i=1}^{3} B_Y\left(y + x_i, \|y - x_i\| + \varepsilon\right) \neq \emptyset, \varepsilon > 0 \text{ in } Y/W \text{with respect to } y + W.$$

Proof. Let $Q_W : Y \to Y/W$ denote the quotient mapping. We consider X/W as a subspace of Y/W. Let $\varepsilon > 0$ and $(u_i)_{i=1}^{3} \subset X/W$. Choose $x_i \in X$ and such that $Q_W\left(x_i\right) = u_i$ for $i = 1, 2, 3$. Since $W \subset X$, we may assume that:

$$\|y - x_i\| < \|Q_W\left(y - x_i\right)\| + \varepsilon = \|Q_W(y) - u_i\| + \varepsilon$$

Choose $x \in X \cap_{i=1}^{3} B_Y\left(y + x_i, \|y - x_i\| + \varepsilon\right)$. Then

$$Q_W(x) \in X/W \cap_{i=1}^{3} B\left(Q_W(y) + u_i, \|Q_W(y) - u_i\| + \varepsilon\right)$$

as desired. □

We state and prove one of our main results in the sequel.

Theorem 8.4. *Let X be a closed subspace of a Banach space Y and let $y \in Y \backslash X$ and $Z = \operatorname{span}(X, \{y\})$.Then the following statements are equivalent.*

(i) X *is a u-ideal in* Z

(ii) $X^{\perp\perp} \cap (\cap_{x \in X} B_{z^{**}}(y + x, \|y - x\|)) \neq \phi$.

(iii) $X \cap (\cap_{i=1}^{n} B_z(y + x_i, \|y - x_i\| + \varepsilon)) \neq \phi$ *for every finite collection* $(x_i)_{i=1}^{n} \subset X$ *and* $\varepsilon > 0$.

(iv) $X \cap (\cap_{i=1}^{3} B_z(y + x_i, \|y - x_i\| + \varepsilon)) \neq \phi$ *for every finite collection of three points* $(x_i)_{i=1}^{3} \subset X$ *and* $\varepsilon > 0$.

Proof. $(i) \Rightarrow (ii)$: Let $T : Z \to X^{**}$ be the operator associated with u-ideal projection. For $x \in X$ we have

$$\|y - x\| = \|y - x - 2{i_x}^{**}T(y - x)\| = \|y + x - 2{i_x}^{**}T(y)\|$$

This means that $2{i_x}^{**}T(y) \in B(y + x, \|y - x\|)$.
(ii) $\Rightarrow$ (iii). Let $x^{***} \in X^{\perp\perp} \cap \cap_{x \in X} B_{z^{**}}(y + x, \|y - x\|) \neq \emptyset$. We use the principle of local reflexivity with $E = \operatorname{span}(i_X^{**}(x^{**}), y, x_1, x_2, \ldots, x_n) \subset Z^{**}$ and $= X^{\perp} \subset Z^{*}$.
$(iii) \Rightarrow (iv)$. Since $i = 1, 2, 3$ gives the three points of (iv).
(iv) $\Rightarrow$ (i). Let X be a finite co-dimensional subspace then it sufficient to show that X/W is a u-ideal in the quotient X/W has property (iv) in Z/W. This reduces the problem to one which is finite-dimensional. Let $r_y = d(Q_W(y), X/W)$ and let $Q_W : Z \to Z/W$ be the quotient mapping. By finite dimensionality there is at least one exposed point $e_0 \in P_{Q_W(y)}$ with exposing functional $e^* \in (X/W)^*$. Let $M = e^*(e_0) = \max_{PQ_W(y)} e^*(e)$ and find $e_1 \in P_{Q_W(y)}$ such that

$$m = e^*(e_1) = \min_{P_{Q_W(y)}} e^*(e). \text{ Choose}$$

$$2c \in X/W \cap B(Q_W(y) + e_0, r_y) \cap B(Q_W(y) + e_1, r_y).$$

We get $2c - e_1 \in P_y$ for $i = 0, 1$ and

$$M \geq e^*(2c - e_1) = 2e^*(c) - m$$

$$m \leq e^*(2c - e_0) = 2e^*(c) - M$$

So that $e^*(c) = \frac{M+m}{2}$ and $M = e^*(2c - e_1)$. Since e_0 is exposed by e^* we get $c = \frac{e_0+e_1}{2}$ and c is also unique. By assumption we have

$$\{2c\} = \bigcap_{i=0,1} B(Q_W(y) + e_i, r_y) \cap B(Q_W(y) + u, \|Q_W(y) - u\|)$$

for all $u \in X/W$ and this means that

$$\|Q_W(y) + u - 2c\| \leq \|Q_W(y) - u\| \;\forall\; u \in X/W.$$

So by the hereditary properties between spaces and their quotients, $X/W \Delta_u Z/W$ in the algebraic structures as required. □

In the proposition that follows we prove general results about centres of symmetry in the compact convex sets.

Proposition 8.2. *Let K be a convex compact set in a locally convex Hausdorff vector space X.Given for every finite-dimensional Banach space Y and every continuous linear operator $T : X \to Y$, $T(K)$ has a center of symmetry then, K has a center of symmetry.*

Proof. For Y a finite-dimensional Banach space and $T : X \to Y$ a continuous linear operator, define $C_{Y,T} = \{x \in K : T(x) \text{ is a center of symmetry in } T(K)\}$. Every $C_{Y,T}$ is non-empty, convex and compact.By taking a finite sum of finite dimensional Banach spaces, we see that the family $C_{Y,T}$ has the finite intersection property.By compactness, there exists $c \in \bigcap_{Y,T} C_{Y,T}$ which is a center of symmetry in K.Indeed, let $x \in K$ and assume $2c - x \notin K$.Then there exists an $x^* \in X^*$ such that $x^*(2c - x) > \sup_{u \in K} x^*(u)$, but this contradicts $c \in C_{\mathbb{R},x^*}$. □

9 Co-Dimension Two

We make the following simple observation in form of a lemma

Lemma 9.1. *Let Y be a Banach space and suppose X and Z are subspaces of Y such that $X \subseteq Z$ and denote a natural embeddings by $i_X : X \to Y, i_Z : Z \to Y$ and $j : X \to Z$. For $y \in Z$ and $x^{**} \in X^{**}$ we have:*

$$\|y - i_X^{**}(x^{**})\| = \|y - j^{**}(x^{**})\|.$$

In particular, the set $P_y \subset X^{**}$ is the same whether it is defined relative to Y or Z. Moreover, $d\left(y, X^{\perp\perp}\right) = d(y, X)$.

Proof. We have $i_X = i_Z j$. Let $x^{**} \in X^{**}, y \in Z \backslash X$ and $y^* \in Y^*$. We have

$$\langle y - i_X{}^{**}(x^{**}), y^* \rangle = \langle i_z(y) - i_z^{**} j^{**}(x^{**}), y^* \rangle = \langle y - j^{**}(x^{**}), i_z^*(y^*) \rangle$$

and it follows that

$$\|y - i_X^{**}(x^{**})\| = \|y - j^{**}(x^{**})\|.$$

We get $d_Y\left(y, X^{\perp\perp}\right) = d_Z\left(y, X^{\perp\perp}\right)$ and by using the principle of local reflexivity in $Z = \operatorname{span}(X, \{y\})$ we find $d_z\left(y, X^{\perp\perp}\right) = d_z(y, X)$ and thus $d_z(y, X) = d_Y(y, X)$. □

The next key result in the foregoing is as follows:

Theorem 9.2. *Let X be a closed subspace of a Banach space Y. If X is an $u-$ ideal in Z for every subspace Z of Y with $\dim Z/X \leq 2$, then X is an u-ideal in Y.*

Proof. We have a possible non-linear $T : Y \to X^{**}$ with $T(x) = x$ for all $x \in X$ such that $\|y - 2i_X{}^{**}T(y)\| = \|y\|$ for all $y \in Y$. For all $y \in Y$, we have that $T(y)$ is a centre of symmetry in P_y. Let $y_1, y_2 \in Y$. Let $Z = \operatorname{span}(X, \{y_1, y_2\})$. By assumption X is a u-ideal in Z which means that T is linear: $T(y_1 + y_2) = \underline{T}(y_1) + T(y_2)$ □

10 Strict u-Ideals in Banach Spaces

We look at strict u-ideals in Banach spaces that is ideals for which the Hahn -Banach extension operator is both strict and unconditional. The main aim being an expansion and extension of the research of [12] related. A Banach space X is a strict ideal $u-$ ideals in its bidual when the canonical decomposition $X^{***} = X^* \otimes Z^\perp$ is unconditional. Godfrey, Kalton and Saphar [5] observed that the theory of u-ideals is much less satisfactory than in the complex case of h - ideal. Under this we endeavor to fill some gaps in the theory of u-ideals which are strict.

10.1 Strict u-Ideals in their Bidual

We use standard Banach space notations. For a Banach space space X, B_X is the closed unit ball and S_X is the unit sphere. The canonical embedding $X \to X^{**}$ is denoted by k_X.

Remark 10.1. l_1 is a $u-$ ideal because it is a $u-$ summand in $l_1{}^{**}$ hence it is not a strict $u-$ ideal.

The theorem that follows extends this remark considerably

Theorem 10.1. *Suppose that X is a separable Banach containing l_1. Let P be a contractive projection on X^{***} with $kerP = X^\perp$ and such that $V = P\left(X^{***}\right)$ is norming. Then $\|I - P\| \geq 2$. Then X cannot be a strict u-ideal.*

Proof. Since V is norming the associated operator $T : X^{**} \rightarrow X^{**}$ is an isometry. If X contains a copy of l_1, then there exists $x^{**} \in X^{**}$ with $\|x^{**}\| = 1$ and such that $\|x^{**} + x\| = \|x^{**} - x\|$ for all $x \in X$. If $\|I - P\| = a$ so that we can find a net (x_d) in X, converging weak * to Tx^{**}, with

$$\limsup \|Tx^{**} - x_d\| \leq a.$$

Since T is an isometry limsup $\|x^{**} - x_d\| \leq a$ and thus

$$\limsup \|x^{**} + x_d\| = \limsup \|Tx^{**} + x_d\| \leq a.$$

However, $\limsup \|Tx^{**} + x_d\| \geq 2$ □

Proposition 10.1. *Let X be either a separable Banach space or a Banach space containing no copy of l_1.*

1. *X is a strict ideal if and only if $\|I - 2\pi\| = 1$ that is if and only if $2 \in G(X)$.*
2. *(If X is complex) X is a strict $h-$ ideal in X^{**} if and only if $\|I - (1 + \lambda)\pi\| = 1$ whenever $|\lambda| \leq 1$ i.e if and only if $G(X) = \{1 + \lambda : |\lambda| \leq 1\}$.*
3. *If X is a strict u-ideal (respectively h-ideal) then every subspace of a quotient space of X Is also a strict u-ideal (respectively h-ideal).*

Definition 10.1. For every Banach space X we let the natural embedding $k_{X^*} : X^* \rightarrow X^{***}$ be an element of $HB\,(X, X^{**})$. And further let $\pi : X^{***} \rightarrow X^{***}$ denote the associated ideal projection with $\ker \pi = X^{\perp}$. We provide the lemma that follows which is key to the subsequent work.

Lemma 10.2. *Let X be a Banach space containing no copy of l_1. Then $\|I - 2\pi\| \leq k_u(X)$*

Proof. Suppose $x^{**} \in S_{X^{**}}$ then $x^{**} \in Ba(X)$ and so, for $\varepsilon > 0$ arbitrary taken then, there is a series $\sum x_k = x^{**}$ weak * such that for all $-1 \leq \theta_k \leq 1$ and all n

$$\left\| \sum_{k=1}^{n} \theta_k x_k \right\| \leq \liminf_{m \rightarrow \infty} \left\| \sum_{k=n+1}^{m} x_k - \sum_{k=1}^{n} x_k \right\| \leq k_u(X) + \varepsilon.$$

Hence if

$$s_n = \sum_{k=1}^{n} x_k \text{ then } \|x^{**} - 2s_n\| \leq k_u\left(x^{**}\right) + \varepsilon.$$

It thus follows that $\|I - 2\pi\| \leq k_u(X) + \varepsilon$. Since $\varepsilon > 0$, was arbitrary taken $\varepsilon \rightarrow 0$ hence we have our desired result. □

The Theorems that follow characterizes spaces which are strict u-ideal in their biduals.

Theorem 10.3. *Let X be a Banach space containing no copy of l_1. Then X is a strict u-ideal if and only if $k_u(X) = 1$.*

Proof. Suppose X is a strict u-ideal in X^{**}. Then the projection $P = \pi$ and the associated operator $T : X^{**} \rightarrow X^{**}$ is the identity. Now if $x^* \in Ba(X)$, then let us select a sequence (x_n) converging weak* to x^{**}. Let further A_n be the convex hull of $\{x_k : k \geq n\}$. If H_n is the weak *-closure of A_n then $\cap_n H_n = \{x^{**}\}$ so that we conclude that

$$k_u\left(x^{**}\right) = \|x^{**}\| = 1.$$

Conversely we suppose that $k_u(X) = 1$ then this direction holds. □

Suppose i_X is the natural embedding $i_X : X \rightarrow Y \cdot P_\phi = \phi \circ i_X^*$ is a norm one projection on Y^* with $\ker P = X^{\perp}$. We say X is an ideal in Y if and only if $HB(X, Y) \neq \emptyset$. When we have $\left\|x^{\perp} + \phi\left(x^*\right)\right\| = \left\|x^{\perp} - \phi\left(x^*\right)\right\|$ for all $x^{\perp} \in X^{\perp}$ and $x^* \in X^*$ we say that X is a $u-$ ideal in Y and ϕ is unconditional. We further note that ϕ is unconditional if and only if $\|I - 2P_\phi\| = 1$. Then we have the well-known notion of $M-$ideal whose work is extensive that is,

$$\left\|x^{\perp} + \phi\left(x^*\right)\right\| = \left\|x^{\perp}\right\| + \left\|\phi\left(x^*\right)\right\| \text{ for all } x^{\perp} \in X^{\perp} \text{ and } x^* \in X^*.$$

Definition 10.2. We say an operator $T_\phi : Y \to X^{**}$ is a norm one operator if

$$\langle i_X^* y^*, T_\phi(y)\rangle = \langle y, P_\phi(y^*)\rangle \text{ for all } y \in Y \text{ and } y^* \in Y^*.$$

By this definition it follows that $T_\phi(x) = x$ for all $x \in X$. X is a strict ideal in Y if there is a $\phi \in HB(X, Y)$ such that $\phi(X^*)$ is norming. In this case ϕ is called strict. Further since $|\langle x^*, T_\phi(y)\rangle| \leq \|T_\phi\| \|x^*\| \|y\|$ we see that ϕ is strict if and only if $T_\phi : Y \to X^{**}$ is isometric.

Proposition 10.2. *Suppose X is a strict u-ideal in Y. Then X is a strict $u-$ ideal in Y if and only if X is a strict u-ideal in span $(X, \{y\})$ for all $y \in Y$.*

Theorem 10.4. *X is a strict $u-$ ideal in X^{**} if and only if $\|I - 2\pi\| = 1$.*

Proof. Assume that X is a strict $u-$ ideal in its bidual. Let $x^{**} \in X^{**} \backslash X$. We have $X \cap (\cap_{x \in X} B_{X^{**}}(x, \|x - x^{**}\|)) = \phi$ since any element in the intersection would define a norm one projection from span $(X, \{x^{**}\})$ onto X which is a contradiction. We thus get $\cap_{x \in X^{**}}(x, \|x - x^{**}\|) = \{x^{**}\}$ which implies that the only element in $HB(X, X^{**})$ is k_{X^*} Since X^* is norming for X^{**} the other direction is trivial. □

The Theorem that follows was first inspired by theorem 5.5 in Godefry [5]. Vegard lima and Asvald Lima [18] in Theorem 2.9 removed the assumption that the space does not contain l_1 as an improvement. We further show that strict u-ideals are separably determined.

Proposition 10.3. *Let X be a Banach space which is a strict u-ideal in its bidual. If Y is any separable subspace of X then it is a strict u - ideal in its bidual and is X separably determined.*

Proof. Let Y be a closed subspace of X with natural embedding $i_X : Y \to X$. Assume the ideal property $\|I - 2\pi_X\| = 1$ where $\pi_X = k_X {K_X}^*$. We need to show that $\|I - 2\pi_Y\| = 1$ where $\pi_Y = k_{Y^*} {K_Y}^*$. Indeed ${i_Y}^{**} k_Y = k_X i_Y$ and ${i_Y}^{***} k_{X^*} = k_Y * {i_Y}^*$ so that ${i_Y}^{***} \pi_X = \pi_Y {i_Y}^{***}$. Now we get

$$1 \geq \|{i_Y}^{***}(I - 2\pi_X)\| = \|(I - 2\pi_Y)\, {i_Y}^{***}\|.$$

Since ${i_Y}^{**} : Y^{**} \to X^{**}$ is isometric, it is onto Y^{**} hence $\|I - 2\pi_Y\| = 1$. Therefore, strict u-ideals are separably determined. □

11 Conclusion

This work has given an account on the ideals in Banach spaces where the various characterizations of $L, M, u, h-$ideals and their variants have been determined up to a classification. Here, classes of: compact operators, bounded linear operators, finite ranks operators in relation to their ideal properties have been exhibited. The general properties of the M-ideals have been studied; key and relevant results presented. In particular, the interplay among the ideals mentioned as well as their extensions have been established. The results demonstrate the existence of classes of closed operator ideals as well as their boundedness in view of Radon-Nickodym properties. Additionally, the findings give the characteristics of ideals through the Hahn-Banach extension opertors, a characterization of L, $M-$ ideals using the standard projections. Moreover the work focussed on $u-$ideals in their biduals with a keen analogy on separability and norm attainability.

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Authors have declared that no competing interests exist.

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