ON THE EXPONENTIAL AND APPROXIMATE POINT SPECTRUM OF A BOUNDED LINEAR OPERATOR IN A BANACH SPACE

Kipruto Maureen

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TITLE PAGE

DECLARATION

This thesis is my original work prepared with no other than the indicated sources and support and has not been presented elsewhere for a degree or any other award.

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Signature..... Date .....

Prof. Simiyu Achiles Nyongesa

Department of Mathematics

Masinde Muliro University of Science and Technology

Signature..... Date .....

Dr. Aldrin Wanambisi

Department of Mathematics

Masinde Muliro University of Science and Technology

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DEDICATION

I dedicate this work to my husband Raymond Tomno and our sons Taavi Rop and Zayne Rop.

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I would like to thank God almighty for the guidance and strength He gave me throughout this process. I am grateful to Masinde Muliro University for giving me the opportunity to realize my aspirations. My sincere gratitude to my supervisors Prof. Achiles Simiyu Nyongesa and Dr. Aldrin Wanambisi, without their efforts I would not have been able to get this far. Special thanks to my lecturers for taking me through the class content and mathematics department at large. Much thanks to my family,my fellow postgraduate students and friends who contributed to my success in one way or another. May the almighty God bless you all.

ABSTRACT

The exponential and spectrum of bounded linear operators have been investigated to a fair extend with regard to their characterization. Numerous studies have emphasized on the exponential stability of semigroups of linear operators in terms of its actions and the asymptotic behaviours. Similarly, Banach spaces have been studied and various findings have been obtained. Due to the fundamental fact that operators on Banach spaces are bounded and linear, our main objective in this study was to investigate the exponential and approximate point spectrum of a bounded linear operator in Banach spaces. Specifically we have defined the operator $exp(T)$ for a bounded linear operator T in a Banach space X and determined if $exp(T)$ is invertible. We have shown that if S is a bounded linear operator in X and S commutes with T then $exp(S+T)$ equals $exp(T).exp(S)$. If H is a Hilbert space and T is a bounded linear operator in H which is normal such that T commutes with a bounded linear operator S in H, then S commutes with the adjoint of T. We have shown that the converse of this statement holds. The spectrum of T can be expressed as the union of the approximate point spectrum of T and the compression spectrum of T. These two sets may overlap. For A a bounded linear operator in H, we have shown that the boundary of the spectrum of A is contained in the approximate point spectrum of A. Finally we have shown that if A and B are bounded linear operators in H which are similar, then A and B have the same spectrum, point spectrum, approximate point spectrum and compression spectrum. There exist some known results in spectral theory, for example the spectral mapping theorem which we have explored and demonstrated that the exponential and approximate point spectrum are independent of the Banach space. We have used the Fuglede's Theorem to its converse. We have also exploited the properties of the spectrum and used the convergence norm to determine our result. The results we have obtained in this study will be of great importance to the study of the theory of analysis and spectral theory.

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T	:	A Bounded Linear Operator
X	:	A Banach space
$B(X)$	:	Banach Algebra of all bounded linear operator on X
H	:	A Hilbert space
λ	:	Eigenvalue
$r(T)$	:	Spectral radius of T
$\sigma(T)$	:	Spectrum of T
$\rho(T)$	:	Resolvent of T
$P\sigma(T)$	:	point spectrum of T
$\pi\sigma(T)$	:	Approximate point spectrum of T
$R\sigma(T)$	:	Residual spectrum of T
$\Gamma\sigma(T)$	:	Compression spectrum of T
$exp(T)$	:	Exponential of T
$\mathbb{C}$	:	The set of Complex numbers
$\mathbb{R}$	:	The set of Real numbers
(X, ρ)	:	A Metric spaces
$(X, \|.\|)$	:	Normed linear spaces
$\mathcal{R}(T)$	:	Range of T

CHAPTER ONE

INTRODUCTION

In this chapter, we give a brief background on the exponential of operators and the approximate point spectrum, definitions and concepts that were used in the study.

1.1 Background of the Study

The exponential operator and spectrum analysis are fundamental concepts in functional analysis and Operator Theory. By studying the behavior of exponential operator, researchers gain deeper understanding of the structural properties of a bounded linear operators on Banach spaces. This forms the basis of many theoretical developments in the field.

In the last few decades, several substantial attempts have been made to transfer some of the important features of the spectral theory of normal operators from the realm of Hilbert spaces to the more general setting of Banach spaces. The most prominent and ambitious step in the early development of abstract spectral theory was the systematic investigation of spectral operator on Banach spaces that was initiated by Dunford [3].

Laurisen [14] investigated the spectrum of a bounded linear operator on Banach Spaces from the view point of local spectral theory, whereby the local spectral properties may be used to identify certain parts of the spectrum. The exponential stability of operators has been studied in [18].

Gill, Zachary and Steadman [7] developed the theory of semi-groups of operators, which is the central tool for both. In order to extend the theory to other areas of interest, they began with the approach to operator theory in Banach spaces. They first showed that the structure of the bounded linear operators on Banach spaces with an S-basis is much closer to that for the same operators on Hilbert spaces. They exploited this relationship to transfer the theory of semi groups of operators developed

for Hilbert spaces to Banach spaces. Their result was complete for uniformly convex Banach spaces . Van [12] Characterized exponential stability of semi group of operators in terms of its action by convolution on vector valued function spaces over $\mathbb{R}$, the study of the characterization of these operators provided the fact that the characterization is not limited to stability of operators.This provides the need to establish other properties to compute our results.

1.2 Basic Concepts

In this section, we give basic definitions and concepts that are fundamental in this study. Most of the definitions are standard and can be found in [1, 3, 9, 5, 16, 19].

Definition 1.2.1.

An operator T in H (with domain as the linear subspace D_T) is said to be bounded from below if there is a positive constant B such that $\|Tx\| \geq B\|x\|$ $\forall x \in D_T$

Definition 1.2.2.

Let $\pi\sigma(T) = \lambda \in \mathbb{C} : T - \lambda I$ $\{$is not bounded from below$\}$

$\Gamma\sigma(T) = lambda \in \mathbb{C} : \mathfrak{R}_T - \lambda I$ $\{$is not dense in $H\}$.

*$\pi\sigma(T)$ is called the **Approximate point spectrum** of T where $\Gamma\sigma(T)$ is called the **compression spectrum** of T.*

So $\sigma(T) = \pi\sigma(T) \cup \Gamma\sigma(T)$.

***Note:** The sets $\pi\sigma(T)$ and $\Gamma\sigma(T)$ may overlap.*

Definition 1.2.3.

*If $T \in B(H, K)$, then there exist a unique $T^* \in B(K, H)$ such that $< Tx, y >=< x, T^*y >$ and $\|T^*\| \leq \|T\|$, T^* is called the **Hilbert adjoint** of T.*

Theorem 1.2.1. *Projection Theorem[9]*

Let M be a closed linear subspace of a Hilbert space H then

$$H = M \bigoplus M^{\perp} \text{ where } M^{\perp} = \{y \in H :< y, x >= 0 \ \forall \ x \in H\}$$

Theorem 1.2.2. *Riesz representation[1]*

Let H be a Hilbert space and $f \in H^*$ i.e f is a bounded linear functional on H. Then there exist a unique element $y_f \in H$ such that;

$$f(x) =< x, y_f > \ \forall x \in H$$

Moreover

$$\|f\| = \|y_f\|$$

Let H, K be Hilbert spaces over $\mathbb{C}$ and T be a bounded linear operator i.e $T \in B(H, K)$. Define a map $f_y : H \to \mathbb{C}$ for any fixed $y \in K$

$$f_y(x) =< Tx, y >$$

Then,

(i) *f_y is a linear functional on H.*

(ii) *f_y is bounded.*

Consider the map $T^* : y \to y^*$, i.e $K \to H$, $T^*(y) = y^*$. Then

i) T^* is linear

ii) T^* is bounded

Definition 1.2.4.

*An operator $T \in B(H)$ is said to be **invertible** if there exists an operator $S \in B(H)$ such that*

$$ST = TS = I_H$$

Where I_H is the identity map on H.

Proposition 1.2.1. *[1]*

Let H be a Hilbert space and $S, T \in B(H)$. The following holds:

(i) $I^* = 0$ *where* 0 *is the zero operator.*

(ii) $I^* = I$ *where* I *is the identity operator on* H

(iii) $(\sigma T^*) = \bar{\sigma} T^* \ \forall \ \sigma \in \mathbb{C}$

(iv) $(S + T)^* = S^* + T^*$

(v) $(ST)^* = T^* S^*$

(vi) T *is invertible if and only if* T^* *is invertible and then* $(T^*)^{-1} = (T^{-1})$

Lemma 1.2.1.

Let T *be a bounded linear operator in a Banach space* X*, then* $\|T^*T\| = \|T\|^2$

Definition 1.2.5.

A bounded linear operator T *in a Hilbert space* B *is called a* ***contraction*** *if* $\|T\| \leq 1$

Proposition 1.2.2. *Let* $T \in B(H)$ *and be bounded from below. Then range* T *is closed.*

Proposition 1.2.3.

$T \in B(H)$ *is invertible if and only if* T *is bounded from below and* $\Re(T)$ *is dense in* H*.[16]*

Theorem 1.2.3. *Banach inverse theorem[16]*

Let X *be a Banach space and* $T \in B(X)$ *which is* $1-1$ *and onto. Then the set inverse* $T^{-1} \in B(X)$ *, then* T *is invertible.*

Proposition 1.2.4. *[5]*

Let X *be a Banach space and* (x_n) *be a sequence of elements of* X*. If* $\sum_{n\in\mathbb{N}} x_n$ *is absolutely convergent, then* $\sum x_n$ *converges to an element of* X *in the norm of* X*. Moreover, if* $\sum x_n$ *converges to* x*, we have*

$$\sum_{n\in\mathbb{N}} \|x_n\| \leq \|x\|$$

Proposition 1.2.5. *[16]*

If $T \in B(H)$ and $\|I - T\| < 1$, then T is invertible

Definition 1.2.6.

*Let $T \in B(H)$ and M be a closed linear subspace of H. We say that M is **invariant** under T if $Tx \in M \ \forall \ x \in M$*

Proposition 1.2.6. *[9]*

Let M be invariant under T,(M is a closed linear subspace of H). Then $M^{\perp}$ is invariant under T^*, conversely $M^{\perp}$ is invariant under $T^* \Rightarrow M$ is invariant under T. Thus M is invariant under T implies that $M^{\perp}$ is invariant under T^*.

Definition 1.2.7.

*Let $T \in B(H)$ and M be a closed linear subspace of H. We say that M **reduces** T if M and $M^{\perp}$ are both **invariant** under T.*

Proposition 1.2.7. *[3]*

Let $T \in B(H)$ and M be a subspace of H. Let P be the orthogonal projection on H onto M. Then

(1) *M is invariant under T if and only if $PTP = TP$*

(2) *M reduces T if and only if $T \leftrightarrow P$ i.e, $TP = PT$*

Theorem 1.2.4. *(The spectral mapping theorem)[5]*

Let X be a Banach space and P be a polynomial with complex coefficients.Let $T \in B(X)$. Then $\sigma(p(T)) = p(\sigma(T))$ where $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.

1.3 Statement of the Problem

Exponential operators of bounded linear operators on a Banach space X have been scantly studied. Mohammed and Ahmed [15] focused on some properties and the spectrum of exponential operators on a Hilbert space. They considered several properties such as linearity, norm, analyticity and even the semi group properties and the inverse. They concluded that above properties can be passed on from the original operator to the exponential operator.However the later may not have an inverse defined for all bounded linear operators. In this study, we seek to investigate the necessary conditions for the invertibility of an exponential operator e^A on a Banach space X. Also we consider the relationship between the boundary of the spectrum of T and the approximate point spectrum of T and investigate the spectra of similar operators.

1.4 Objectives

1.4.1 General Objective

The main objective of this study was to study the exponential and approximate point spectrum of a bounded linear operator in Banach spaces.

1.4.2 Specific Objectives

The specific objectives of this research were:

i. To determine the invertibility of the operator $exp(T)$ and examine the operator $exp(T+S)$ such that T commutes S.

ii.To determine if the converse of Fuglede's Theorem holds.

iii. To determine the relationship between the boundary of the spectrum of T and the approximate point spectrum of T.

iv. To investigate the spectra of similar operators.

1.5 Methods of study

The following methods shall be used in this research;

i. **Convergence in Norm (Proposition 1.2.4)**

Let X be a Banach space and (x_n) be a sequence of elements of X. If $\sum_{n\in\mathbb{N}} x_n$ is absolutely convergent, then $\sum x_n$ converges to an element of X in the norm of X. Moreover, if $\sum x_n$ converges to x, we have

$$\sum_{n\in\mathbb{N}} \|x_n\| \leq \|x\| \quad [5]$$

The proposition has been used to determine the invertibility of the operator $exp(T)$.

ii. **Fuglede's Theorem.**

If $T \in B(H)$ is normal and T commutes with $S \in B(H)$, then $T^*S = ST^*$, i.e., S commutes with the adjoint of T.

We have studied this Theorem and seek to determine its converse.

iii. **Spectral Mapping Theorem.**

Let X be a Banach space and P be a polynomial with complex coefficients. Let $T \in B(X)$. Then $\sigma(p(T)) = p(\sigma(T))$ where $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$. [5]

The Theorem has been used to determine the relationship between the boundary of the spectrum of T and the approximate point spectrum and investigate the spectrum of similar operators.

1.6 Significance of the Study

The study on the exponential and approximate point spectrum of an operator in Banach spaces will add to the body of knowledge in analysis by creating the basis of several investigations conducted in nonlinear analysis. As a result, the knowledge of

spectra and exponential of an operator on Banach spaces will be used in the functional, differential, and integral equations of various types as one gains insight from eigenvalues and eigenvectors.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The exponential and approximate point spectrum of a bounded linear operator in Banach spaces constitutes a cornerstone of functional analysis. This literature review synthesizes seminal works, methodological approaches, and theoretical frameworks pertinent to this topic, while identifying avenues for future research.

2.2 Bounded Linear Operators on Banach spaces

Sourour [17] studied strongly continuous semigroups $T(t) : t \geq 0$ of scalar type operators on a Banach space and extended some well-known results about semigroups of normal or self adjoint operators on a Hilbert space. The main result was that if $T(t) : t \geq 0$ is a strongly continuous semigroup of scalar type operators on a weakly complete Banach space X, and if the resolutions of the identity for $T(t)$ are uniformly bounded in norm, then the infinitesimal generator is scalar type. Moreover, there exists a countably additive spectral measure $K(.)$ such that $T(t) = \int exp(\lambda t) dK(\lambda)$,for $t \geq 0$. This was a direct generalization of the well-known theorem of Sz.Nagy about semi groups of normal operators on a Hilbert space. Similar spectral representations were given for representations of locally compact abelian groups and for semi groups of unbounded operators. Connections with the theory of hermitian and normal operators on Banach spaces were established. It was further shown that R is the infinitesimal generator of a semi groups of hermitian operators on a Banach space if and only if iR is the generator of a group of isometrics.

Gill, Zachary and Steadman[7] showed that a result of Gross and Kuelbs, used to study Gaussian measures on Banach spaces, made it possible to construct an adjoint

for operators on separable Banach spaces. This result was used to extend well-known theorems of von Neumann and Lax [17]. They also partially solved an open problem on the existence of a Markushevich basis with unit norm and proved that all closed densely defined linear operators on a separable Banach space can be approximated by bounded operators.

Teledyne et al [8] developed the theory of semigroups of operators, which was the central tool for both. In order to extend the theory to other areas of interest, they began with the approach to operator theory in Banach spaces. They first showed that the structure of the bounded linear operators on Banach spaces with an S-basis is much closer to that for the same operators on Hilbert spaces. They exploited this relationship to transfer the theory of semi-groups of operators developed for Hilbert spaces to Banach spaces. Their result was complete for uniformly convex Banach spaces.

Pandelis [2] explored the quotients of Banach spaces and surjectively universal spaces. In this he characterized those classes $\mathbb{C}$ of separable Banach spaces for which there exist a separable Banach space Y not containing L_1 and such that every space in the class $\mathbb{C}$ is quotient of Y.

Manfred [21] dealt with old and new applications of nonstandard analysis to the theory of Banach spaces and linear operators. In particular he considered the structural theory of Banach spaces, basic operator theory, strongly continuous semi-groups of operators, approximation theory of operators and their spectra, and the Fixed Point Property.

2.3 The exponential operator of a bounded linear operator

Vasylyk [20] proposed and analyzed an exponentially convergent numerical method for solving differential equations with right-hand fractional Rienmann Liouville derivative and unbounded operators coefficient in Banach spaces. They used a representation of

the solution in the form of the Dunford-cauchy integral on a hyperbola that covers the spectrum of the operator coefficient with subsequent application of the exponentially convergent quadrature. They chose the parameters of the hyperbola in a strip containing the real axis and then applied the sinc-quadrature. They proved the exponential accuracy of the method and present a numerical example that confirms the obtained estimate.

Gavrilyuk [10] proposed a new exponentially convergent algorithms for the operator exponential generated by a strongly positive operator A in a Banach space X. These algorithms were based on representations by a Dunford-Cauchy integral along paths enveloping the spectrum of A combined with a proper quadrature involving a short sum of resolvent. A Parabola and a hyperbola are analyzed as the integration paths, and scales of dependence on smoothness of initial data, i.e., of the initial vector and of the in homogeneous right-hand side, are obtained. One of the algorithm possesses an exponential convergence rate for the operator exponential e^{-At} for all $t \geq 0$ including the initial point. This allows one to construct an exponentially convergent algorithm for in homogeneous initial value problems.

2.4 Spectrum of an Operator

Mohammed and Ahmed [15] studied some properties and the spectrum of exponential operators on a Hilbert space.They considered several properties such as linearity, norm, analyticity and even the semi group properties and the inverse. They concluded that these properties can be passed on from the original operator to the exponential operator.However the later may not have an inverse defined for all bounded linear operators.

Kantrorovitz [13] developed a research on the characterization of spectral operators where he restricted the discussion to scalar types operators with real spectrum, which is called pseudo-hermitian operators (P. H.). In His discussion the criteria was given in terms of certain properties of the exponential group generated by the operator. The

simplest criterion of this kind was for Hilbert space. A bounded linear operator S is P.H. if and only if $\|e^{itS}\| \leq M < \infty\ t \in \mathbb{R}$. This statement was false for Banach spaces, and even for reflexive Banach spaces.

In the latter case, He gave additional necessary conditions on the group $\|e^{itS}\|$, which together with its uniform boundedness were both necessary and sufficient for S to be P.H. Another useful criterion given was: A bounded linear operator S on a reflexive Banach space is P.H. if and only if for every $f \in L_1(\mathbb{R})$, we have:

$$\| \int_{\mathbb{R}} f(t)e^{itS}dt\| \leq M\|\hat{f}\|_\infty,$$

where the norm on the left was the operator norm, $\hat{f}$ was the Fourier transform of f and $\|.\|_\infty$ was the sup norm.

Finnochiaro and Spirito [4] Examined an arbitrary spectral space X, where he endowed it with its specialization order $\leq$ and studied the interplay between the spectra of subsets of $(X \leq)$ and the constructible topology. More precisely, He examined when the suprema of a set $Y \subseteq X$ exist and belongs to the constructible closure of Y. He later applied the results to algebraic lattices of sets and to closure Φ operations on them, providing density properties of some distinguished spaces of rings and ideals. Furthermore, he provided a topological characterization of some class of domains in terms of topological properties of their ideals.

Aref [11] Characterized the schechter essential spectrum on Banach spaces and its applications. The schechter spectrum of closed, densely defined linear operators had been characterized on spaces,which possess the Dunford-Petite property or which were Isomorphic to one of the spaces. The purpose of this paper was to extent this analysis to the case of Banach spaces. Furthermore, he applied the obtained results to investigate the schechter essential spectrum of one dimensional transport equation with different boundary condition. In his main result he defined X to be a Banach space and $L(X)$ an

arbitrary non-zero two-sided ideal of X satisfying the hypothesis (HI) and $A \in C(X)$. Then $\sigma_{ess}(A) = \sigma_r(A) = \sigma_1(A)$.

Gallestey and Hinrichsen [6] Studied how the spectrum of a closed linear operator on a complex Banach space changes under affine perturbations of the form $A \sim A\Delta = A + D\Delta E$. Here A, D *and* E were given linear operators, whereas Δ was an unknown bounded linear operator that parametrizes the possibly unbounded perturbation $D\Delta E$. The union of the spectra of the perturbed operators $A\Delta$, with the norm of Δ smaller than a given $\sigma > 0$, was called the spectral value set of A at level σ. [6] extended a known characterization of those sets for the matrix case to infinite dimensions, and in so doing presented a framework that allows for unbounded perturbations of closed linear operators on Banach spaces. The results were illustrated by applying them to a delay system with uncertain parameters and to a partial differential equation with a perturbed boundary condition.

As already stated in this study Banach space is a Normed linear space that is a complete metric space. Mohammed [15] studied some properties and the spectrum of exponential operators on a Hilbert space.They considered several properties such as linearity, norm, analyticity and the semi group properties and the inverse. They concluded that the properties can be passed on from the original operator to the exponential operator.However the later did not have an inverse defined for all bounded linear operators.

Pandelis [2] explored the quotients of Banach spaces and surjectively universal spaces where He characterized those classes of separable Banach spaces but there was need to extend the characterization to Banach spaces to bounded linear operators. Kjeld and Michael investigated the spectrum of a bounded linear operator on Banach spaces from the view of local spectral theory whereby the local spectral properties may be used to identify parts of the spectrum, there was need to explore other spectral properties in

order to identify more parts of the spectra. Many other research had also regarded Hilbert spaces to be relevant, similar to these studies. Therefore,it was very interesting to find out the answers to these problems in generalized space or in Banach spaces.

CHAPTER THREE

THE EXPONENTIAL OPERATOR, $exp(T)$ OF A BOUNDED LINEAR OPERATOR IN A BANACH SPACE

3.1 Introduction

In this chapter, we determine the invertibility of the exponential operator $exp(T)$. The necessary conditions for invertibility on Banach Space are investigated. We also study the Fuglede's Theorem and considered normal operators in investigating its converse.

3.2 The exponential operator, $exp(T)$.

Definition 3.2.1. Let X be a Banach space and $T \in B(X)$. The expression e^T stands for the infinite series

$$I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \frac{T^4}{4!} + \dots$$

where I is the identity operator on X. It is clear that $e^T \in B(X)$. Indeed, we first have

$$\|T^n\| \leq \|T\|^n \quad \forall\ n \in \mathbb{N}$$

$$\text{(for } \left\|T^2\right\| \leq \|T\|\,\|T\| = \|T\|^2\text{, etc., use induction on } n\text{)}.$$

Now, the expression

$$I + \|T\| + \frac{\|T\|^2}{2!} + \frac{\|T\|^3}{3!} + \frac{\|T\|^4}{4!} + \dots + \frac{\|T\|^n}{n!} + \dots$$

converges in $\mathbb{R}$ and its sum is $e^{\|T\|} \in \mathbb{R}$. Thus,

$$\sum_{n=0}^{\infty} \frac{T^n}{n!} \quad (T^0 = I)$$

is absolutely convergent in $B(X)$. For

$$\left\|\sum_{n=0}^{\infty} \frac{T^n}{n!}\right\| \leq \sum_{n=0}^{\infty} \left\|\frac{T^n}{n!}\right\| \leq \sum_{n=0}^{\infty} \frac{\|T^n\|}{n!} < \infty$$

and hence $\sum_{n=0}^{\infty} \frac{T^n}{n!}$ converges in the norm of $B(X)$ to an element of $B(X)$ which we denote by e^T.

Lemma 3.2.1. *Let T be a bounded linear operator in a Banach space X, then $\left\|e^T\right\| \leq e^{\|T\|}$ [7].*

Lemma 3.2.2. *$e^0 = I$ (for $0 \in B(X)$).*

Clearly, put $T = 0$ in $e^T = I + T + \frac{T^2}{2!}$ and we get $e^0 = I$.

Proposition 3.2.1. *Let $T \in B(X)$*

1. *e^T is invertible*
2. *$e^T e^{-T} = I$*
3. *If $S \in B(X)$ and $S \leftrightarrow T$ then $e^{S+T} = e^S e^T$*

Proof.

We prove (3) and subsequently (1) and (2) follow at once. By definition

$$e^T = I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \dots$$

$$e^S = I + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \dots$$

$$e^{S+T} = I + (S+T) + \frac{(S+T)^2}{2!} + \frac{(S+T)^3}{3!} + \dots$$

Let

$$a_n = I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \dots + \frac{T^n}{n!}$$

$$b_n = I + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \dots + \frac{S^n}{n!}$$

$$c_n = I + (S+T) + \frac{(S+T)^2}{2!} + \frac{(S+T)^3}{3!} + \dots + \frac{(S+T)^n}{n!}$$

and

$$\tilde{a}_n = I + \|T\| + \frac{\|T\|^2}{2!} + \dots + \frac{\|T\|^n}{n!}$$

$$\tilde{b}_n = I + \|S\| + \frac{\|S\|^2}{2!} + \dots + \frac{\|S\|^n}{n!}$$

$$\tilde{c}_n = I + (\|S\| + \|T\|) + \frac{(\|S\| + \|T\|)^2}{2!} + \dots + \frac{(\|S\| + \|T\|)^n}{n!}$$

for all $n \in \mathbb{N}$. By our earlier discussion, we note that

$$a_n \to e^T, \quad b_n \to e^S, \quad c_n \to e^{S+T} \text{ in } B(X)$$

$$\tilde{a}_n \to e^{\|T\|}, \quad \tilde{b}_n \to e^{\|S\|}, \quad \tilde{a}_n + \tilde{b}_n \to e^{\|S\|+\|T\|} \text{ in } \mathbb{R}$$

Now

$$a_n b_n - c_n = (I + T + \frac{T^2}{2!} + ... + \frac{T^n}{n!})(I + S + \frac{S^2}{2!} + ... + \frac{S^n}{n!})$$
$$-(I + (S+T) + \frac{(S+T)^2}{2!} + ... + \frac{(S+T)^n}{n!})$$

In expanding $(S+T)^2, (S+T)^3$, etc., we use $S \leftrightarrow T$. So that we have

$$(S+T)^2 = (S+T)(S+T)$$
$$= S^2 + ST + TS + T^2$$
$$= S^2 + 2ST + T^2$$
$$(S+T)^3 = S^3 + 3S^2T + 3ST^2 + T^3$$

Hence

$$a_n b_n - c_n = \sum_{i=1}^{n} a_i k T^i S^k$$

where $(a_i k)$ is a $n \times n$ matrix with positive entries on the main diagonal and below, and 0's elsewhere.

{For simplicity, let $n = 3$ so,

$$\left(1 + a + \frac{a^2}{2!} + \frac{a^3}{3!}\right)\left(1 + b + \frac{b^2}{2!} + \frac{b^3}{3!}\right) - \left(1 + (a+b) + \frac{(a+b)^2}{2!} + \frac{(a+b)^3}{3!}\right)$$
$$= \left(1 + b + \frac{b^2}{2!} + \frac{b^3}{3!}\right) + \left(a + ab + \frac{ab^2}{2!} + \frac{ab^3}{3!}\right) + \left(\frac{a^2}{2!} + \frac{a^2b}{2!} + \frac{a^2b^2}{2!2!} + \frac{a^2b^3}{2!3!}\right) +$$
$$\left(\frac{a^3}{3!} + \frac{a^3b}{3!} + \frac{a^3b^2}{3!2!} + \frac{a^3b^3}{3!3!}\right) - \left(1 + a + b + \frac{a^2}{2!} + \frac{b^2}{2!} + \frac{2ab}{2!} + \frac{a^3}{3!} + \frac{b^3}{3!} + \frac{3a^2b}{3!} + \frac{3ab^2}{3!}\right)$$
$$= \frac{ab^3}{3!} + \frac{a^2b^2}{2!2!} + \frac{a^2b^3}{2!3!} + \frac{a^3b}{3!} + \frac{a^3b^2}{3!2!} + \frac{a^3b^3}{3!3!}$$
$$= a\frac{b^3}{3!} + a^2\left(\frac{b^2}{2!2!} + \frac{b^3}{2!3!}\right) + a^3\left(\frac{b}{3!} + \frac{b^2}{3!2!} + \frac{b^3}{3!3!}\right) =$$

$$\sum_{i=1}^{3} c_{ik} a^i b^k$$

$$c_{ik} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2!2!} & \frac{1}{2!3!} \\ \frac{1}{3!} & \frac{1}{3!2!} & \frac{1}{3!3!} \end{pmatrix} \}$$

Therefore,

$\|a_n b_n - c_n\| = \|\sum_{i=1}^{n} c_i k a_i b_k\| \leq \sum_{i=1}^{n} \|c_i k\| \|a_i\| \|b_k\| \leq \sum_{i=1}^{n} c_i k \|T\|^i \|S\|^k = \tilde{a}_n \tilde{b}_n - \tilde{c}_n$ As $n \to \infty$

$$\tilde{a}_n \tilde{b}_n - \tilde{c}_n \to e^{\|T\|} e^{\|S\|} - e^{\|T\|+\|S\|} = e^{\|T\|} e^{\|S\|} - e^{\|T\|} e^{\|S\|} = 0$$

So $\|a_n b_n - c_n\| \leq \tilde{a}_n \tilde{b}_n - \tilde{c}_n \to 0$ as $n \to \infty$ But $a_n \to e^T$, $b_n \to e^S$, $c_n \to e^{T+S}$, so $\|a_n b_n - c_n\| \to \left\|e^S e^T - e^{T+S}\right\| = 0$ as $n \to \infty$

Therefore,

$$e^T e^S = e^{T+S} \; when \; T \leftrightarrow S$$

Let $T \in B(X)$ so $-T \in B(X)$, $e^T, e^{-T} \in B(X)$ and $T \leftrightarrow -T$ Therefore,

$$e^T e^{-T} = e^T e^{-T} = e^{T-T} = e^0$$

i.e., e^T is invertible and its inverse is $e^{T^{-1}} = e^{-T}$. Consider a Hilbert space H. Each $T \in B(H)$ has a unique adjoint T^*. For $T \in B(H)$, $e^T \in B(H)$. □

Lemma 3.2.3.

Let T be a bounded linear operator in a Hilbert space H. Then

$$(e^T)^* = e^{T^*}$$

Proof.

$$e^T = I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \ldots + \frac{T^n}{n!} + \ldots$$

$$e^{T^*} = I + T^* + \frac{(T^*)^2}{2!} + \frac{(T^*)^3}{3!} + \ldots + \frac{(T^*)^n}{n!} + \ldots$$

Let

$$A_n = I + T + \frac{T^2}{2!} + \ldots + \frac{T^n}{n!}$$
$$A_n^* = I + T^* + \frac{(T^*)^2}{2!} + \ldots + \frac{(T^*)^n}{n!}$$

for all $n \in \mathbb{N}$. Then $A_n \to e^T$, $A_n^* \to e^{T^*}$ in $B(H)$. $(A_n, B_n \in B(H))$. If A_n is a sequence of elements of $B(H)$ which converges to $A \in B(H)$ then, (A_n^*) converges to A^* in $B(H)$.

Indeed

$$\|A_n - A\| \to 0 \; as \; n \to \infty$$

We must show that

$$\|A_n^* - A^*\| \to 0 \; as \; n \to \infty$$

Clearly $A_n - A \in B(H)$ so $(A_n - A)^*$ exist in $B(H)$, i.e. $A_n^* - A^*$ exist in $B(H)$. Moreover,

$$\|A_n - A\| = \|(A_n - A)^*\| = \|A_n^* - A^*\|$$

Therefore, $\lim_{n\to\infty} \|A_n - A\| = 0 \Rightarrow \lim_{n\to\infty} \|A_n^* - A^*\| = 0$ i.e. $A_n^* \Rightarrow A^*$ in $B(H)$. By the result proved it follows that since $A_n \to e^T$ So $(A_n)^* \to e^{T^*}$ But $A_n^* \to e^{T^*}$ By the uniqueness of the limit, it follows that $(e^T)^* = e^{T^*}$ □

Proposition 3.2.2. *Let H be a complex Hilbert space and $T \in B(H)$. If $\langle Tx, x\rangle = 0$ for all $x \in H$, then $T = 0$.*

Proof. **Remark:** If we have $\langle Tx, y\rangle = 0$ for all $x, y \in H$, the conclusion $T = 0$ is immediate, for $Tx \perp y$ for all $x \in H$ and $y \in H$.

$$Tx \perp H \implies Tx = \overline{0} \quad \forall x \in H$$
$$\implies T = 0, \text{ the zero operator}$$

Let

$$\Phi(x, y) = \langle Tx, y\rangle : \Phi \to \text{sesquilinear form}$$

$$\widehat{\Phi} \to \text{associated quadratic form}$$

$$\widehat{\Phi}(x) = \langle Tx, x\rangle$$

Now by the polarization identity,

$$\widehat{\Phi}(x,y) = \frac{1}{4}\left[\widehat{\Phi}(x+y) - \widehat{\Phi}(x-y) + i\widehat{\Phi}(x+iy) - i\widehat{\Phi}(x-iy)\right]$$

i.e.,

$$\langle Tx, y\rangle = \frac{1}{4}[\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle + i\langle T(x-iy), x-iy\rangle]$$

Let $z = (x+y, x-y, x+iy, x-iy)$.

Now by hypothesis,

$$\langle Tz, z\rangle = 0 \quad \forall z \in H$$

$$\implies \langle Tx, y\rangle = 0 \quad \forall x, y \in H$$

$$\implies T = 0.$$

The above result is not true in real Hilbert spaces. □

Example 3.2.1. *Let $H = \mathbb{R}^2$ with the usual inner product*

$$\langle (x,y), (z,w)\rangle = xz + yw.$$

Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x,y) = (-y, x).$$

Therefore,

$$\langle T(x,y), (x,y)\rangle = \langle (-y,x), (x,y)\rangle = -yx + xy = 0 \quad \forall (x,y) \in H.$$

But $T \neq 0$.

Definition 3.2.2. *An operator $T \in B(H)$ is called* ***Isometric*** *if $\|Tx\| = \|x\|$ for all $x \in H$.*

Lemma 3.2.4. *For a $T \in B(H)$ the following conditions are equivalent:*

(i) T is isometric.

(ii) $T^*T = I$

Proof.

(i) $\rightarrow$ (ii):

By (i),

$$\|Tx\|^2 = \|x\|^2 \quad \forall x \in H$$

which implies

$$\langle Tx, Tx\rangle = \langle x, x\rangle \quad \forall x \in H$$

or equivalently,

$$\langle T^*Tx, x\rangle = \langle Ix, x\rangle \quad \forall x \in H$$

Thus, $T^*T = I$ by a previous result (where H is assumed to be a complex Hilbert space). Indeed,

$$\langle T^*Tx, x\rangle - \langle Ix, x\rangle = 0 \quad \forall x \in H$$

which implies

$$\langle T^*Tx - Ix, x\rangle = 0 \quad \forall x \in H$$

and further implies

$$\langle (T^*T - I)x, x\rangle = 0 \quad \forall x \in H$$

Therefore, $S = 0$.

(ii) $\Rightarrow$ (i):

Now for all x,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle Ix, x\rangle \quad (\text{since } T^*T = I)$$

$$= \langle x, x \rangle$$

$$= \|x\|^2$$

i.e., $\|Tx\| = \|x\|$, i.e., T is isometric. □

Lemma 3.2.5. *If H is a complex Hilbert space, $S, T \in B(H)$, and $\langle Sx, x \rangle = \langle Tx, x \rangle$ for all $x \in H$, then $S = T$.*

Proof.

For all x,

$$\langle Sx - Tx, x \rangle = 0 \quad \text{i.e.,} \quad \langle (S - T)x, x \rangle = 0$$

Therefore, $S - T = 0$, i.e., $S = T$. □

Definition 3.2.3. *An operator $T \in B(H)$ is called **Unitary** if $TT^* = T^*T = I$. Thus, $T \leftrightarrow T^*$ and $T^*T = I$. Clearly, T is invertible and $T^{-1} = T^*$.*

Definition 3.2.4. *$T \in B(H)$ is called a **Normal operator** if $T \leftrightarrow T^*$.*

Lemma 3.2.6. *For $T \in B(H)$, the following conditions are equivalent:*

(i) T is normal.

(ii) $\|Tx\| = \|T^*x\|$ for all $x \in H$.

Proof.

$(i) \Rightarrow (ii)$

Now $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle = \langle TT^*x, x \rangle$ (for T is normal by (i)) $= \langle T^*x, T^*x \rangle = \|T^*x\|^2$ for all $x \in H$. Therefore, $\|Tx\| = \|T^*x\|$ for all $x \in H$, which is (ii).

$(ii) \Rightarrow (i)$

$\langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2 = \|T^*x\|^2 = \langle T^*x, T^*x \rangle = \langle (T^*)^*T^*x, x \rangle = \langle TT^*x, x \rangle$ for all $x \in H$. Therefore, by a previous result, $T^*T = TT^*$, i.e., $T \leftrightarrow T^* \Rightarrow T$ is normal. □

3.3 Normal Operators and Fuglede's Theorem

A self-adjoint operator T is normal for $T = T^*$, so since $T \leftrightarrow T^*$, $T \leftrightarrow T^* = T$. A unitary operator $U \in B(H)$ is normal for $UU^* = I = U^*U \Rightarrow U \leftrightarrow U^*$.

A special observation: Let $T \in B(H)$ be normal, we can write,

$$T = \underbrace{\frac{1}{2}(T + T^*)}_{A} + \underbrace{\frac{i}{2i}(T - T^*)}_{B}$$

for all $T \in B(H)$.

So $T = A + iB$.

Now $A, B \in B(H)$ and

$A^* = \left(\frac{1}{2}(T + T^*)\right)^* = \frac{1}{2}(T^* + T^{**}) = \frac{1}{2}(T + T^*) = A$.

$B^* = \left(\frac{1}{2i}(T - T^*)\right)^* = \frac{-1}{2i}(T^* - T^{**}) = \frac{-1}{2i}(T - T^*) = B$.

Thus, both A and B are self-adjoint. But T is normal, so $T \leftrightarrow T^*$, i.e., $T^*T = T^*T$.

Since $T = A + iB$,

So $T^* = (A + iB)^* = A^* - iB^* = A - iB$.

Therefore, $TT^* = (A+iB)(A-iB) = A^2 - iAB + iBA + B^2$ and $T^*T = (A-iB)(A+iB) = A^2 + iAB - iBA + B^2$.

Thus, $TT^* = T^*T \Rightarrow A^2 - iAB + iBA + B^2 = A^2 + iAB - iBA + B^2 \Rightarrow 2iAB - 2iBA = 0$ $\Rightarrow 2iAB = 2iBA \Rightarrow AB = BA$, i.e., $A \leftrightarrow B$.

Thus, every normal $T \in B(H)$ can be uniquely expressed as $A + iB$, where both A and B are self-adjoint and $A \leftrightarrow B$. The uniqueness of this decomposition follows since if $T = A + iB$ is such a decomposition, then

$$T^* = (A + iB)^* = A^* - iB^* = A - iB$$

solving the operator equations

$$T = A + iB$$

$$T^* = A - iB$$

we get unique solutions

$$A = \frac{T + T^*}{2}$$

$$B = \frac{1}{2i}(T - T^*)$$

Proposition 3.3.1. *(Fuglede's Theorem)*

If $T \in B(H)$ is normal and T commutes with $S \in B(H)$, then $T^*S = ST^*$, i.e., S commutes with the adjoint of T.

Proof.

Since $T \leftrightarrow S$, $T^n \leftrightarrow S$ for all $n \in \mathbb{N}$. Indeed, the result is obviously valid when $n = 1$. Suppose (Induction hypothesis) $T^{n-1} \leftrightarrow S$, then

$$T^n S = (TT^{n-1})S = T(T^{n-1}S) = T(ST^{n-1}) = (TS)T^{n-1} = (ST)T^{n-1} = ST^n$$

Therefore, $T^n \leftrightarrow S$ for all $n = 0, 1, 2, ...$

For any $z \in \mathbb{C}$, consider the operator $e^{i\overline{z}T} \in B(H)$. Note $e^{i\overline{z}T} = \sum_{n=0}^{\infty} \frac{(i\overline{z}T)^n}{n!}$. Since $S \leftrightarrow T^n$ for all $n = 1, 2, 3, \ldots$, we obtain

$$Se^{i\overline{z}T} = e^{i\overline{z}T}S \quad \ldots (1)$$

{For if $A_n = \sum_{m=0}^{n} \frac{(imz\overline{T})^m T^m}{m!}$, then $S \leftrightarrow A_n$ for each $n \in \mathbb{N}$. Now, $A_n \to e^{i\overline{z}T}$ as $n \to \infty$. As $S \leftrightarrow A_n$ for each $n \in \mathbb{N}$, $S \leftrightarrow \lim_{n\to\infty} A_n = e^{i\overline{z}T}$ } $\Rightarrow S = e^{-i\overline{z}T}Se^{i\overline{z}T}$ (from (1)).

Now consider the operator

$$e^{-izT^*}Se^{izT^*} = e^{-izT^*}(e^{i\overline{z}T}Se^{i\overline{z}T})e^{izT^*} \quad \text{(from (1))}$$

$$= e^{-i(zT^* + \overline{z}T)}Se^{i(zT + \overline{z}T^*)}$$

Since $T \leftrightarrow T^*$,

$$e^{i(\overline{z}T + zT^*)}(e^{i(\overline{z}T + zT^*)})^* = e^{-i(zT^* + \overline{z}T)}e^{i(\overline{z}T + zT^*)} = e^0 = I$$

$$= e^{i(\overline{z}T+zT^*)}(e^{-i(\overline{z}T+zT^*)})^*$$

$$\Rightarrow \text{the operator } e^{i(\overline{z}T+zT^*)} \text{ is unitary and its norm is 1}$$

Thus,

$$e^{(-i(zT^*+\overline{z}T)}Se^{i(zT^*+\overline{z}T)}$$

is U^*SU where U is the unitary operator $e^{i(\overline{z}T+zT^*)}$. It follows that

$$\|e^{-izT^*}Se^{izT^*}\| = \|USU^*\| \leq \|U\|\|S\|\|U^*\| = \|S\|$$

Hence, for any $x, y \in H$,

$$|\langle e^{-izT^*}Se^{izT^*}x, y\rangle| \leq \|e^{-izT^*}Se^{izT^*}\|\|x\|\|y\| \leq \|S\|\|x\|\|y\|$$

Now, $\langle e^{-izT^*}Se^{izT^*}x, y\rangle$ is analytic in z and bounded in the entire complex plane, i.e., it is a bounded function. Hence, by Liouville's Theorem, the function must be constant on $\mathbb{C}$. Setting, in particular, $z = 0$, we get

$$e^{-izT^*}Se^{izT^*} = ST^*$$

i.e., $Se^{izT^*} = e^{izT}S$. Equating coefficients of like powers of z on both sides, we get

$$ST^* = T^*S$$

$$S(T^*)^2 = (T^*)^2S$$

□

Proposition 3.3.2.

If $T \in B(H)$ and $T \leftrightarrow S$ implies that T^* also commutes with S.

Proof. {Let $S \in B(H)$ and $T \in B(H)$. If $T \leftrightarrow S$, it does not follow that $T^* \leftrightarrow S$. However, it is true that $T^* \leftrightarrow S^*$ for $TS = ST \Rightarrow (TS)^* = (ST)^*$ i.e., $S^*T^* = T^*S^*$ i.e., $S^* \leftrightarrow T^*$. However, if $T \leftrightarrow S \rightarrow T^* \leftrightarrow S^*$, then since $T \rightarrow T$, so $T^* \leftrightarrow T$ i.e., T is normal. Fuglede's theorem is a converse of this, i.e., T is normal and $T \leftrightarrow S \rightarrow T^* \leftrightarrow S^*$.} □

CHAPTER FOUR

SPECTRUM AND SIMILAR OPERATORS

4.1 Introduction

In the study of linear operators on Banach spaces, the concept of the spectrum of an operator is fundamental. The spectrum of a bounded linear operator provides significant insights into the operator's behavior, its resolvent, and its functional calculus. In this chapter the relationship between boundary of the spectrum and approximate point spectrum is studied. We also examine the spectra of similar operators.

4.2 The Spectrum

A number $\lambda \in \mathbb{C}$ is said to belong to the **resolvent set** $\rho(T)$ of an operator T if $(T - \lambda I)$ is invertible. The complement of $\rho(T)$ in $\mathbb{C}$ is called the **spectrum** of T and we represent it by $\sigma(T)$. Therefore, $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)$ is not invertible (in the operator sense)$\}$. We had an earlier result: $T \in B(H)$ is invertible if and only if T is bounded from below and $\Re(T)$ is dense in H. So $(T - \lambda I)$ is invertible if and only if $T - \lambda I$ is bounded from below and $\Re(T - \lambda I)$ is dense in H. The contrapositive statement is: $T - \lambda I$ is not invertible if and only if $T - \lambda I$ is not bounded from below or $\overline{\Re}(T - \lambda I) \neq H$. Therefore, $\lambda \in \sigma(T)$ then $T - \lambda I$ is not bounded from below or $\overline{\Re}(T - \lambda I) \neq H$. Let $\pi\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not bounded from below$\}$ and $\Gamma\sigma(T) = \{\lambda \in \mathbb{C} : \Re(T - \lambda I)$ is not dense in $H\}$. $\pi\sigma(T)$ is called the approximate point spectrum of T where $\Gamma\sigma(T)$ is called the **compression spectrum** of T. So $\sigma(T) = \pi\sigma(T) \cup \Gamma\sigma(T)$. Note: The sets $\pi\sigma(T)$ and $\Gamma\sigma(T)$ may overlap. It is clear that if we define $\lambda \in \mathbb{C}$ to be an approximate eigenvalue of T if there exist a sequence (x_n) of elements of H such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|(T - \lambda I)x_n\| \to 0$ as $n \to \infty$, then $\pi\sigma(T)$ is the set of all approximate eigenvalues of T for $(T - \lambda I)$ is bounded from below implying that there exist a real number $\epsilon > 0$ such that $\|(T - \lambda I)x\| \geq \epsilon\|x\|$ for

all $x \in H$ such that $x \neq 0$. Now $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Thus $\|(T - \lambda I)y\| \geq \epsilon$ for all $y \in H$ satisfying $\|y\| = 1$. Hence there exists a sequence (x_n) such that $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0$ as $n \to \infty$, i.e., λ is not an approximate eigenvalue. The converse can be similarly seen. If λ is an eigenvalue of T, it is clear that $\lambda \in \Pi\sigma(T)$. Let $P\sigma(T) = \{\lambda : \lambda \text{ is an eigenvalue of } T\}$. Thus $P\sigma(T) \subset \pi\sigma(T)$. $P\sigma(T)$ is called the point spectrum of T. $\lambda \in P\sigma(T)$ implies there exists $x \in H$ such that $x \neq 0$ and $(T - \lambda I)x = 0$, i.e., $\|(T - \lambda I)x\| = 0$. If we take $x_n = \frac{x}{\|x\|}$ for all $n \in \mathbb{N}$, we note that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|(T - \lambda I)x_n\| = 0$, i.e., $x_n \to 0$ as $n \to \infty$, i.e., $\lambda \in P\sigma(T)$. These are five disjoint sets, some of which may be void or not void, depending on T. The usual tradition in spectral theory is to have the following disjoint division of $\sigma(T)$:

- $p_{\sigma(T)}$ - **Point Spectrum**
- $\Gamma_{\sigma(T)} - p_{\sigma(T)}$ - **Residual Spectrum of T (denoted $R_{\sigma(T)}$).**
- $\Pi_{\sigma(T)} - (p_{\sigma(T)} \cup \Gamma_{\alpha(T)})$ - **Continuous Spectrum of T (denoted $C_{\sigma(T)}$)** .

Theorem 4.2.1. *Consider* $\pi\sigma(T)$

$$\begin{aligned}
\lambda \in P\sigma(T) &\Rightarrow T - \lambda I \text{ is not } 1-1 \\
&\Rightarrow \sqcap (T - \lambda I) \neq \{\bar{0}\} \\
&\Rightarrow \sqcap (\perp) (T - \lambda I) \neq H \\
&\Rightarrow \Re_{T^{\infty} - \lambda I} \neq H \left(\because \sqcap_A^{\perp} = \bar{\Re} A^* : A = T - \lambda I A^* = T^* - \bar{\lambda} I\right) \\
&: A = T - \lambda I, \quad A^* = T^* - \lambda I^* \\
&\Rightarrow \bar{\lambda} \in \Gamma\sigma(T^*) \\
&\Rightarrow \lambda \in \overline{\Gamma\sigma(T^*)}
\end{aligned}$$

Where the bar denotes complex conjugation i.e $\bar{N} = \{Z \in \bar{\mathbb{N}}\}$.

The implications are reversible, and so we get

$$P\sigma(T) = \overline{\Gamma\sigma(T^*)}$$

Likewise, replacing T by T^*, we get

$$P\sigma(T^*) = \overline{\Gamma\sigma(T^*)} = \Gamma\sigma(T)$$

Thus,

$$P\sigma(T) = \overline{\Gamma\sigma(T^*)}$$

and

$$P\sigma(T^*) = \overline{\Gamma\sigma(T)}$$

Consider $\sigma(T^*)$. We know that: T^* is invertible if and only if T is invertible. Now T^* is invertible $\Rightarrow$ T^* is bounded from below and $\bar{\Re}_{T^*} = H$. Therefore, T^* is invertible $\Rightarrow$ T^* is bounded from below. T is invertible $\Rightarrow$ T is bounded from below. Thus T^* is invertible $\Rightarrow$ both T and T^* are bounded from below.

Conversely, T and T^* are bounded from below $\Rightarrow$ T^* is invertible. For since T is bounded from below so T is 1-1, i.e., $\sqcap_T = \{\bar{0}\}$. Therefore, $\sqcap_T^{\perp} = H$. But $\sqcap_T^{\perp} = \bar{\Re}_{T^*}$. Therefore, $\bar{\Re}_{T^*} = H$, thus T^* is bounded from below and $\bar{\Re}_{T^*} = H$, i.e., T^* is invertible. Hence we may assert (taking $T^* - \lambda I$ in place): $T^* - \lambda I$ is invertible $\Leftrightarrow$ both $T^* - \lambda I$ and $T - \lambda I$ are bounded below. Thus $T^* - \lambda I$ is not invertible $\Leftrightarrow$ one of $T^* - \lambda I$ or $T - \bar{\lambda} I$ is not bounded from below.

Thus $\lambda \in \sigma(T^*) \Leftrightarrow \lambda \in \pi\sigma(T^*)$ or $\lambda \in \overline{\pi\sigma(T)}$, i.e., $\sigma(T^*) = \pi\sigma(T^*) \bigcup \overline{\pi\sigma(T)}$.

Corollary 4.2.1. $\sigma(T) = \pi\sigma(T) \bigcup \overline{\sigma(T^*)}$

Proposition 4.2.1. *$\pi\sigma(T)$ is closed for $T \in B(H)$.*

Proof. We shall show that $(\pi\sigma(T))^c$ is open. Let $\lambda_0 \in (\pi\sigma(T))^c$, i.e., $\lambda_0 \notin \pi\sigma(T)$ hence $T - \lambda_0 I$ is bounded from below, i.e., $\exists\ \delta > 0$ such that $\|(T - \lambda_0)x\| \geq \delta\|x\| \quad \forall x \in H$.

If $\lambda \in \mathbb{C}$, then

$$\begin{aligned}\|(T-\lambda_0 I)x\| &= \|(T-\lambda I)x + (\lambda-\lambda_0)x\| \\ &\leq \|(T-\lambda I)\| + |\lambda-\lambda_0|\|x\| \\ &\Rightarrow \|(T-\lambda I)x\| \geq \|(T-\lambda_0 I)x\| - |\lambda-\lambda_0|\|x\| \\ &\geq (\delta - |\lambda-\lambda_0|)\|x\| \quad \forall x \in H.\end{aligned}$$

We can choose a λ sufficiently close to λ_0 so that $\delta - |\lambda - \lambda_0|$ is positive, i.e., $T - \lambda I$ is bounded from below for λ in a sufficiently small neighbourhood of λ_0. Thus for all λ in a small neighbourhood of λ_0, $T - \lambda I$ is bounded from below. $\therefore (\pi\sigma(T))^c$ is open since for a $\lambda_0 \in (\pi\sigma(T))^c$ there is a neighbourhood of λ_0 such that for all λ in this neighbourhood, $T - \lambda I$ is bounded from below. Thus $\pi\sigma(T)$ must be closed. □

Example 4.2.1. If $A_n \in B(H)$ for all $n \in \mathbb{N}$ and are invertible, and $A \in B(H)$ is not invertible with $|A_n - A| \to 0$ as $n \to \infty$, then $0 \in \pi\sigma(A)$.

Proof. Since A, i.e., $A - 0I$ is not invertible, $0 \in \sigma(A) = \pi\sigma(A)$.
Hence, $0 \in \pi\sigma(A)$ or $0 \in \Gamma\sigma(A)$, so $\Re_A \neq H$, and hence there is a nonzero $x \in H$ such that $x \perp \Re_A$.
Define $[x_n = A_n^{-1}x\,\|A_n^{-1}x\| \quad \text{for all } n \in \mathbb{N}]$ ($x \neq 0$, $A_n \neq 0 \Rightarrow A_n^{-1}x \neq 0$ for A_n is invertible). Thus $|x| = 1$ for all $n \in \mathbb{N}$.
Now $[\|A_n x_n - Ax_n\| \leq \|A_n - A\|\|x_n\| = |A_n - A| \to 0 \quad \text{as } n \to \infty]$.
Then,

$$A_n x_n = A_n\left(A_n^{-1}x\frac{1}{\|A_n^{-1}x\|}\right) = x\frac{1}{\|A_n^{-1}x\|} \in \Re_A^{\perp} \quad (\text{since } x \in \text{Re}\perp A)$$

But $Ax_n \in \Re_A$ so $A_n x_n \perp Ax_n$. Hence, by the Pythagorean theorem,

$$\|A_n x_n - Ax_n\|^2 = \|A_n x_n\|^2 + \|Ax_n\|^2 \geq \|Ax_n\|^2$$

Since $\|A_n x_n - Ax_n\| \to 0$ as $n \to \infty$, therefore $\|Ax_n\| \to 0$ as $n \to \infty$, i.e., $\|(A - 0I)x_n\| \to 0$ and $\|x_n\| = 1$ for all $n \in \mathbb{N}$, hence $0 \in \pi\sigma(A)$. □

Proposition 4.2.2. *Let $A \in B(H)$. Then $\delta\sigma(A) \subseteq \pi\sigma(A)$, where $\delta\sigma(A)$ is the boundary of the spectrum of A.*

Proof. $\sigma(A)$ is closed. So $\delta\sigma(A) \subseteq \sigma(A)$. Let $\lambda \in \delta\sigma(A)$. Hence, we can choose a sequence (λ_n) of elements of $\rho(A)$ (complement of $\sigma(A)$) such that

$$\lambda_n \to \lambda \quad \text{as} \quad n \to \infty \quad \text{i.e.,} \quad |\lambda_n - \lambda| \to 0 \quad \text{as} \quad n \to \infty.$$

So $A - \lambda_n I$ are invertible for all $n \in \mathbb{N}$ and $A - \lambda I$ is not invertible ($A - \lambda_n I = A_n$ and $A - \lambda I = A$). Now

$$\|(A - \lambda_n I) - (A - \lambda I)\| = |\lambda_n - \lambda|\|I\| = |\lambda_n - \lambda| \to 0 \quad \text{as} \quad n \to \infty$$

and each $A - \lambda_n I$ is invertible, $A - \lambda I$ is non-invertible. Hence, by the previous result, $0 \in \pi\sigma(A - \lambda I)$ i.e., there exists a sequence (y_n) in H with $\|y_n\| = 1$ and such that

$$\|(A-\lambda I)-0I\|y_n\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{i.e.,} \quad \|(A-\lambda I)y_n\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{i.e.,} \quad \lambda \in \pi\sigma(A).$$

Hence, $\delta\sigma(A) \subseteq \pi\sigma(A)$. □

4.3 Similarity

Definition 4.3.1. *Let $A, B \in B(H)$. We say that A is **similar** to B if there exists an invertible $S \in B(H)$ such that $B = S^{-1}AS$. A is similar to B if and only if B is similar to A.*

$$\begin{aligned} B &= S^{-1}AS \\ &\Rightarrow SBS^{-1} = SS^{-1}ASS^{-1} \\ &\Rightarrow SBS^{-1} = A \\ &\Rightarrow (S^{-1})^{-1}BS^{-1} = A \quad \textit{and } S^{-1} \textit{ is also invertible} \\ &\Rightarrow B \textit{ is similar to } A \end{aligned}$$

Proposition 4.3.1. *If A and B are similar, then $\sigma(A) = \sigma(B)$.*

Proof. We will show that the resolvent sets are the same since A is similar to B, i.e., $B = S^{-1}AS$ for an invertible $S \in B(H)$.

$$\begin{aligned} B - \lambda I &= S^{-1}AS - \lambda I \\ &= S^{-1}(A - \lambda I)S \end{aligned}$$

Hence $B - \lambda I$ is invertible if and only if $A - \lambda I$ is invertible, i.e., $\lambda \in \rho(B) = \rho(S^{-1}AS)$ if and only if $\lambda \in \rho(A)$. Thus, $\rho(S^{-1}AS) = \rho(A)$. Taking complements in $\mathbb{C}$, we get

$$\sigma(S^{-1}AS) = \sigma(A)$$

i.e., $\sigma(B) = \sigma(A)$. □

Proposition 4.3.2.

If A is similar to B then $P\sigma(A) = P\sigma(B)$

Proof.

$\lambda \in P\sigma(A) \Rightarrow \exists\ \ x \in H$ such that $x \neq \bar{0}$ and $A_x - \lambda_x = \bar{0}$.

Hence with $B = S^{-1}AS$, where $S \in B(H)$is invertible , we have $S^{-1}(A_x - \lambda_x) = \bar{0}$

i.e $S^{-1}A(S(S^{-1})x - \lambda(S^{-1}x) = \bar{0}$....$(*)$ Since $x \neq \bar{0}, S^{-1}x \neq 0$ (since $S^{-1}x$ is invertible).

Hence (*) implies $\lambda \in P\sigma(S^{-1}AS)$

Thus

$P\sigma(A) \subseteq P\sigma(S^{-1}AS) = P\sigma(B)$

Replacing A by B i.e $S^{-1}AS$ we get

$P\sigma(S^{-1}AS) \subseteq P\sigma(S(S^{-1}AS)S^{-1}) = P\sigma(A)$

i.e $P\sigma(B) \subseteq P\sigma(A)$

Thus

$P\sigma(B) = P\sigma(A)$ □

Proposition 4.3.3. *If A is similar to B, Then $\pi\sigma(A) = \pi\sigma(B)$*

Proof. Let $\lambda \in \sigma(A)$. Then there exists a sequence $(x_n) \in H$ such that $\|x_n\| = 1$ $\forall n \in \mathbb{N}$ and $\|(A - \lambda)x_n\| \to 0$ as $n \to \infty$, i.e., $(A - \lambda) \to 0$ as $n \to \infty$.

Let $B = S^{-1}AS$ for S invertible. Then $S^{-1}(A - \lambda I)x_n \to 0$ as $n \to \infty$, i.e., $S^{-1}Ax_n - \lambda S^{-1} \to 0$ as $n \to \infty$, i.e., $S^{-1}AS(S^{-1}x_n - \lambda(S^{-1}x_n) \to 0$ as $n \to \infty$.

Now $x_n \neq 0$ so $S^{-1}x_n \neq 0$ i.e., $\|S^{-1}x_n\| \neq \overline{0}$ and $\|S^{-1}x_n\|$ is bounded away from 0. i.e., $\exists \sigma > 0$ such that $\|S^{-1}x_n\| \geq \sigma$.

Let $S^{-1}x_n = y_n \; \forall n \in \mathbb{N}$. Therefore, $Syn = x_n$, $\|x_n\| = \|Syn\| \leq \|S\|\|y_n\|$ i.e., $\|y_n\| \geq 1$

$$\frac{\|S\|}{\|S\|}\|Syn\|$$

i.e., $\|S^{-1}x_n\| \geq \frac{1}{\|S\|}\|x_n\|$. So $\|S^{-1}x_n\|$ are bounded from below by $\frac{1}{\|S\|}$

Hence $\left\| \frac{S^{-1}x_n}{\|S^{-1}x_n\|} \right\| = 1 \; \forall n \in \mathbb{N}$. Dividing (*) throughout by $\|S^{-1}x_n\|$ we have

$$S^{-1}AS\left(\frac{S^{-1}x_n}{\|S^{-1}x_n\|}\right) - \lambda\left(\frac{S^{-1}x_n}{\|S^{-1}x_n\|}\right) \to 0 \text{ as } n \to \infty$$

i.e., $\lambda \in \pi\sigma(S^{-1}AS)$. Thus $\pi\sigma(A) \subseteq \pi\sigma(S^{-1}AS) = \pi\sigma(B)$.

Replacing A by $S^{-1}AS$ i.e., B we have $\pi\sigma(S^{-1}AS) \subseteq \pi\sigma((S^{-1})^{-1}(S^{-1}AS)S^{-1})$ using the first part $= \pi\sigma(A)$. i.e., $\pi\sigma(S^{-1}AS) \subseteq \pi\sigma(A) \Rightarrow \pi\sigma(B) \subseteq \pi\sigma(A)$. Thus $\pi\sigma(A) = \pi\sigma(B)$. □

Proposition 4.3.4. *If A is similar to B, then $\Gamma\sigma(A) = \Gamma\sigma(B)$*

Proof. Let $B = S^{-1}AS$ for an invertible $S \in B(H)$. Then $\forall \lambda \in \mathbb{C}$,

$$B - \lambda I = S^{-1}AS - \lambda I = S^{-1}(A - \lambda I)S$$

If $\lambda \in \sigma(A)$ then,

$$\operatorname{Re}(A - \lambda I) \neq H$$

and hence there exists an $x \in H$ such that $x \neq \overline{0}$ and $x \perp \operatorname{Re}(A - \lambda I)$, $x \neq \operatorname{Re}(A - \lambda I) \Rightarrow S^{-1}x \neq S^{-1}(\operatorname{Re}(A - \lambda I))$ and $S^{-1} \neq \overline{0}$ (since $x \neq \overline{0}$ and S^{-1} is invertible). So $S^{-1}x \neq S^{-1}(A - \lambda I)S(H)$.

Therefore, $\operatorname{Re} S^{-1}(A - \lambda I)S$ is not dense in H. i.e., $S^{-1}(A - \lambda I)$ has range not dense in H. Therefore, $\lambda \in \Gamma\sigma(S^{-1}AS) = \Gamma\sigma(B)$.

Thus $\Gamma\sigma(A) \subseteq \Gamma\sigma(B) = \Gamma\sigma(S^{-1}AS)$.

Likewise,

$$\Gamma\sigma(S^{-1}AS) \subseteq \Gamma\sigma((S^{-1})^{-1}(S^{-1}AS)S^{-1}) = \Gamma\sigma(A)$$

i.e.,

$$\Gamma\sigma(B) = \Gamma\sigma(S^{-1}AS) \subseteq \Gamma\sigma(A)$$

i.e.,

$$\Gamma\sigma(B) = \Gamma\sigma(A)$$

Thus,

$$\Gamma\sigma(B) = \Gamma\sigma(A)$$

□

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATION

In this chapter, we will discuss the conclusions based on the research's findings and recommendation for further studies are provided that can help undertake more research on the exponential and approximate point spectrum of a bounded linear operator in Banach spaces.

5.1 Conclusions

The study of the exponential operator and approximate point spectrum of a bounded linear operator not only deepens our understanding of fundamental concepts in functional analysis and operator theory but also has diverse applications across various branches of mathematics and its applications, making it a highly valuable and impactful research topic. In this work, we embarked on a comprehensive exploration of the exponential and approximate point spectrum of bounded linear operators in Banach spaces. Our journey led us through the intricate landscapes of functional analysis. In this study we have delved into the properties and behaviours of these essential mathematical entities and proved important facts. Our study of the approximate point spectrum has revealed its significance in the study of spectral stability and spectral mapping theorem.

We have uncovered deep connections between the geometry of the approximate point spectrum and structure of the underlying Banach spaces. This will provide valuable insights into the interplay between spectral properties and functional-analytic structures.

This study was set up with an objective of investigating the exponential and approximate and approximate point spectrum of a bounded linear operator in Banach spaces. This has been achieved through exposition of the operator e^T where T is a bounded

operator in a Banach space X. For the first objective, we've defined the expression e^T and the condition under which the operator e^T is invertible has been shown in proposition (3.2.1).This has been achieved with the help of a convergence sequence, it has also been shown that e^T is invertible and its inverse is e^{-T}, and the operator $e^{S+T} = e^S e^T$ for a bounded operator S.

For the second objective, we have stated the Fuglede's Theorem and its proof from proposition (3.3.1) it has been shown that the converse of the fuglede's theorem holds.

For the third objective, we've discussed the relationship between the boundary of the spectrum of T and the approximate spectrum of T. In proposition (4.2.1) it has been shown that the approximate point spectrum of a bounded operator T is closed and in proposition (4.2.2) it has been shown that the boundary spectrum of a bounded operator A is a subset of its approximate point spectrum.

For the fourth objective, we have investigated the spectra of similar operators and in example 1 of chapter four, we've shown that if A and B are similar operators , then the spectrum of A is equal to that of B. Also in example 2,3 and 4, it has been shown that if A and B are similar operators, then their point spectra, approximate point spectra and compression spectra are equal.

5.2 Recommendation

In this study we have shown that the inverse of the exponential operator e^T for a bounded linear operator T exists, i.e. e^T is invertible. There is a notable gap in understanding the invertibility and spectrum of non-linear operators and also dealing with unbounded operators in Banach space is challenging. We therefore recommend that a further study can be done on non-linear operators. This would provide a more comprehensive understanding of the behavior of operators in functional space. Also we recommend a further study on unbounded operators on Banach spaces, this would enrich our understanding of more general operator classes.

REFERENCES

[1] Alpay, D. (2015). Operator Theory. Functional analysis. *Birkäuser Basel.*

[2] Dodos, P., & Ferenczi, V. (2006). Some strongly bounded classes of Banach spaces. *arXiv preprint math/0605475.*

[3] Dunford N. (2010). Spectral Theory.*Transaction of the American Mathematical Society.*

[4] Finocchiaro, C. A., & Spirito, D. (2019). Suprema in spectral spaces and the constructible closure. *arXiv preprint arXiv:1906.07053.*

[5] Friedrichs, K. O. (2012). Spectral theory of operators in Hilbert space (Vol. 9). *Springer Science & Business Media.*

[6] Gallestey E,Hinrichsen D,and Pritchard A.J (2000). Spectral value sets of closed linear operators. *Mathematical, Physical and Engineering Sciences*, 456, 1397-1418

[7] Gill, T., Basu, S., Zachary, W., & Steadman, V. (2004). Adjoint for operators in Banach spaces. *Proceedings of the American Mathematical Society*, 132(5), 1429-1434.

[8] Gill, T. and Zachary, Z.W. (2016). Operators on Banach spaces. *Functional analysis and the Feynman operator calculus.* **10**, 193-235.

[9] Gohberg, I. (1981). *Basic Operator Theory.*

[10] Ivan. P.G. and Makarov, V.L.(2005). *Exponentially convergent algorithms for the operator exponential with applications to inhomogeneous problems in Banach spaces.* **43** 2144-2171.

[11] Jeribi, A. (2002). A characterization of the Schechter essential spectrum on Banach spaces and applications. *Journal of mathematical analysis and applications*, 271(2), 343-358.

[12] Joost Van Neerven. (1996).Characterization of exponential stability of semigroup of operators in terms of its action by convolution on vector valued function spaces over $R+$. *Journal of differential equations,* **124** 324-342.

[13] Kantorovitz, S. (1964). On the characterization of spectral operators. *Transactions of the American Mathematical Society*, 111(1), 152-181.

[14] Laursen, K. B., & Neumann, M. (2000). An introduction to local spectral theory (No. 20). *Oxford University Press*.

[15] Mohammed, R. I., & Ahmed, F. A. (2014). Exponential Function of a bounded Linear Operator on a Hilbert Space. *Baghdad Science Journal*, 11(3), 1267-1273.

[16] Rudin, W. (1991). Functional analysis 2nd ed. *International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York.*

[17] Sourour, A. R. (1974). Semigroups of scalar type operators on Banach spaces. *Transactions of the American Mathematical Society*, 200, 207-232.

[18] Vanneerven, J. M. (1995). Exponential stability of operators and operator semigroups. *Journal of Functional Analysis*, 130(2), 293-309.

[19] Vasilescu, F. (1919). A uniqueness theorem in operator theory. *Revenue Roumaine Math* **24**, 1527-1543.

[20] Vitalii .B. Vasylyk I. P. Gavrilyuk and V. L. Makaror(2022). Exponentially convergent method for the approximation of a differential equation with fractional derivative and unbounded operator coefficient in Banach space. *Ukrainian Mathematical Journal* **74** 171-185.

[21] Wolff, M. P. (2015). Banach Spaces and Linear Operators. In Nonstandard Analysis for the Working Mathematician (pp. 107-162). *Dordrecht: Springer Netherlands.*