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Research Article

Local Fractional Strong Metric Dimension of Certain Complex Networks

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Fractional variants of distance-based parameters have application in the fields of sensor networking, robot navigation, and integer programming problems. Complex networks are exceptional networks which exhibit significant topological features and have become quintessential research area in the field of computer science, biology, and mathematics. Owing to the possibility that many real-world systems can be intelligently modeled and represented as complex networks to examine, administer and comprehend the useful information from these real-world networks. In this paper, local fractional strong metric dimension of certain complex networks is computed. Building blocks of complex networks are considered as the symmetric networks such as cyclic networks C_n, circulant networks $C_n(1,2)$, mobious ladder networks M_{2n}, and generalized prism networks G_m^n. In this regard, it is shown that LSFMD of $C_n(n\geq 3)$ and $G_m^n(n\geq 6)$ is 1 when n is even and $n/n-1$ when n is odd, whereas LSFMD of M_{2n} is 1 when n is odd and $n/n-1$ when n is even. Also, LSFMD of $C_n(1,2)$ is $n/2(\lceil m+1/2\rceil)$ where $n\geq 6$ and $m=\lceil n-5/4\rceil$.

1. Introduction

Distance-based parameters for networks play a vital role in pharmaceutical chemistry [1], network discovery [2], robot navigation, and optimizations [3]. Many real-life large-scale systems having substantial topological features can be modeled as complex networks such as social networks, information networks, technological networks, and biological networks. This representation has innovative impacts to information processing and co-ordination of these large-scale networks. Management of large-scale networks such as Internet with their tremendous growth and heterogeneity is a challenging mathematical problem which have profound implications for the efficient design of future communication networks. Complex networks are composed of building blocks, and if the building blocks are considered as symmetric networks, then complexity of these networks can be reduced for better analysis and interpretation. A few important building blocks are cycles, circulant networks, mobious ladder networks, and generalized prism networks, which are discussed in this article.

Over the past few decades, circulant and mobious ladder networks have been comprehensively explored by many researchers due to their vast application and importance in telecommunication networks [4], computer science (see [5, 6]), chemistry [7], discrete mathematics, and very large-scale integration (VLSI) design. Complex large-scale interconnection networks used in the design of local area networks, distributed computer systems, and telecommunication networks have been constructed based on VLSI circuit technology. In telecommunication networks, many stations are placed at short distances (less than 5 km) to share data at a very high speed, and the main objective is to optimize the exchange of data with an efficient network topology.

In a finite network N of order n, $V(N)$ and $E(N)$ represent the collection of vertices and edges of the network N, respectively. The collection of all the vertices of the network N that are adjacent to the vertex v is known as the open neighbourhood of any vertex v in N. The distance between the vertices v_1 and v_2 of N denoted by $d(v_1,v_2)$ is the length of shortest path (geodesic) between these vertices. A pair of vertices v_1 and v_2 of N is said to be mutually maximally distant if v_1 is maximally distant from v_2 and v_2 is maximally distant from v_1 where the vertex v_1 is maximally distant from v_2 if $d(v_1, v_2) \geq d(v, v_2)$ for all v in the open neighbourhood of v_1. A vertex w of N is said to resolve two vertices v_1 and v_2 of N if v_1 and v_2 are at unequal distance from the vertex w. A set S of vertices of the network N is a resolving set for N if every two distinct vertices of N are resolved by some vertex of S. Metric basis is the resolving set having minimum cardinality, and this cardinality is said to be the metric dimension (MD) of N, denoted by $\dim(N)$. In 1975, the notion of MD was initiated by Slater [8], motivated by the problem of uniquely determining the location of an intruder in a network and later studied independently by Harary and Melter in [9]. MD has been heavily studied, and the advancements in this field can be seen in [10]. Some bounds for MD in terms of diameter of network are given in [1]. Chartrand et al. [1] formulated MD as integer programming problem. The problem of finding MD of a graph is NP-hard (see [11]). The MD of trees and Cayley diagraphs are studied in [1, 12], respectively. A pair of vertices v_1 and v_2 in N is said to be strongly resolved by a vertex v, if there exist either a shortest path from v_1 to v containing v_2 or a shortest path from v_2 to v containing v_1. Strong resolving set S of N is a collection of vertices such that each distinct pair of vertices in N is strongly resolved by some vertex in S. Strong metric basis of N is the strong resolving set having smallest cardinality, and this cardinality is labelled as strong metric dimension (SMD) of N, denoted by $\mathrm{sdim}(N)$. In 2004, SMD of a network was discovered by Sebő and Tannier [13] and later in 2007, and computation of SMD was declared as NP-hard problem by Oellermann and Peters-Fransen [14]. The resolving neighbourhood (RN) denoted by $R\{v_1, v_2\}$ for a pair of vertices v_1 and v_2 in N is composed of all vertices at varying distances from v_1 and v_2. If $\eta\colon V(N) \longrightarrow [0,1]$ is a real valued function that assigns a number between 0 and 1 to each vertex of N and $U \subseteq V(N)$, then the function η applied on the set U is given by $\eta(U) = \sum_{v \in U} \eta(v)$. If the weight of $R\{v_1, v_2\}$ is greater than or equal to 1 for any two vertices $v_1 \neq v_2$ in N, then the function η is called resolving function of N. The fractional metric dimension (FMD) of N expressed as $\dim_f(N)$ is given by the least possible weight of a resolving function of N. In 2001, Currie and Oellermann [10] initiated the concept of FMD by formulating the linear programming problem using the integer programming problem that was presented for MD given in [1]. This relaxation technique transforms an NP-hard integer programming into a related problem that is solvable in polynomial time. In 2012, Arumugam and Mathew [15] defined FMD using the concept of resolving neighbourhoods. In [16], FMD of Generalized Jahangir graph was calculated. In [17], FMD of tree and unicyclic graphs was computed. FMD of hierarchical product, corona product, and lexicographic product graphs were calculated in [18, 19]. The problem of computing the FMD for all the connected networks is an NP-hard problem. Strong resolving neighbourhood (SRN) denoted by $S\{v_1, v_2\}$ for the pair of vertices v_1 and v_2 in N is the set of all vertices $w \in V(N)$ such that either v_1 lies on $w - v_2$ geodesic or v_2 lies on $w - v_1$ geodesic. If the weight $\eta(S\{v_1, v_2\})$ is greater than or equal to 1, then the real-valued function that assigns a number between 0 and 1 to each vertex of N given by $\eta\colon V(N) \longrightarrow [0,1]$ is known as a strong resolving function of N for each distinct pair of vertices in $V(N)$. The fractional strong metric dimension (FSMD) of N expressed as $\mathrm{sdim}_f(N)$ is given by the least possible weight of a strong resolving function of N. In 2013, Kang and Yi [20] gave the notion of FSMD, studied it for various significant finite connected graph classes and mentioned that FSMD problem can be interpreted as linear programming problem with the same strategy as in [12]. In 2010, Okamoto et al. [21] gave the concept of local metric dimension (LMD) by considering the adjacent vertices of graph only. A set of vertices W in a connected network N is a local metric set of N if every two adjacent vertices of N are distinguished by some vertex of W. Local metric basis is the local metric set having smallest cardinality, and this cardinality is said to be the LMD of N, denoted by $\mathrm{ldi\,m}(N)$. In [22, 23], LMD of corona product graphs and circulant graphs has been discussed, respectively. LMD of some families of graphs was given in [24, 25]. The local resolving neighbourhood (LRN) denoted by $L\{v_1, v_2\}$ for a pair of adjacent vertices v_1 and v_2 in N is composed of all vertices which are resolved by $L\{v_1, v_2\}$. The concept of local resolving neighbourhood and local resolving function arises similar to resolving neighbourhood and resolving function in case of dealing with only the pair of adjacent vertices. In [26], authors set forth a localized variant of FMD known as local fractional metric dimension (LFMD) and studied it for strong and cartesian products of graphs. LFMD of the network N denoted by $\mathrm{ldi\,m_f}(N)$ is the least possible weight of local resolving function of N. LFMD of rotationally symmetric planar graphs arisen from planar chorded cycles was computed in [27]. In [28, 29], LFMD of rotationally symmetric and planer networks and corona products graphs were computed, respectively. Local strong resolving neighbourhood (LSRN) $L_S\{v_1, v_2\}$ for the pair of adjacent vertices v_1 and v_2 in N is the set of all vertices $w \in V(N)$ such that either v_1 lies on $w - v_2$ geodesic or v_2 lies on $w - v_1$ geodesic. If for each adjacent pair of vertices in $V(N)$, the weight $\alpha(L_S\{u_1, u_2\})$ is greater or equal to 1, then the mapping $\alpha\colon V(N) \longrightarrow [0,1]$ is called a local strong resolving function of N, where $\alpha(L_S(x,y)) = \sum_{x \in L_S(x,y)} \alpha(x)$. The local fractional strong metric dimension (LFSMD) of the network N denoted by $\mathrm{lsdim}_f(N)$ is defined as the least possible weight of a local strong resolving function of N. In [30], the notion of LFSMD was introduced, and the authors devised a combinatorial technique to compute LFSMD of a general network and was further applied to compute LFSMD for rotationally symmetric and planer networks. In [30], the notion of LFSMD was initiated. The combinatorial criteria to calculate LFSMD of a general network was devised and further applied to compute LFSMD for rotationally symmetric and planer networks. This criteria is given in Lemma 1. This motivated us to compute LFSMD of certain complex networks with symmetric networks as their building blocks. The symmetric networks considered in this article are cyclic networks C_n, circulant networks $C_n(1,2)$, mobious ladder networks M_{2n}, and generalized prism networks G_m^n. The collection of LSRNs of a network N with

least cardinality and its compliment is represented by the notations $\mathscr{L}(N)$ and $\overline{\mathscr{L}}(N)$, respectively. Here, $\mathscr{L}(N) = \{LS(N) \mid LS(N)$ is th
e LSRN with the condition $|LS(N)| = \gamma(N)\}$, where $\gamma(N)$ is the cardinality of smallest SRNs of N. Moreover, $\overline{\mathscr{L}}(N) = \{\tilde{L} \mid \tilde{L}$ is the LSRN of N not in $\mathscr{L}(N)\}$.

Lemma 1 (see [30]). *Let $\mu(N) = \mathscr{L}(N) \cup \overline{\mathscr{L}}(N)$ be a set consisting of all LSRNs of network N in such a manner that for every adjacent pair of vertices x and y in the vertex set of N, if the condition $|S\{x, y\} \cap [\cup LS(N)]| \geq \gamma(N)$ holds. Then,* $\text{lsdim}_f(N) = \sum_{s=1}^{\beta(N)} (1/\gamma(N))$, where $\beta(N) = |[\cup LS(N)]|$.

1.1. Main Results. The research conducted in this article leads to the following results:

Theorem 1

(1) For $n \geq 3$, $\text{lsdim}_f(C_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 2; \\ n/n-1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$

(2) For $n \geq 6$,

(a) $\text{lsdim}_f(C_n(1,2)) = n/2(\lceil m + 1/2 \rceil)$

(b) $\text{lsdim}_f(M_{2n}) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 2; \\ n/n-1 & \text{if } n \equiv 0 \pmod 2 \end{cases}$

(c) $\text{lsdim}_f(G_m^n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 2; \\ n/n-1 & \text{if } n \equiv 1 \pmod 2. \end{cases}$

The remaining part of the article is structured in the following manner. Sections 2 and 3 are devoted for LSRNs and LFSMD of certain complex networks with symmetric building blocks.

2. Local Strong Resolving Neighbourhoods of Certain Complex Networks

In this section, we compute LSRNs of certain complex networks. These complex networks are composed of building blocks to unravel the dynamics of these networks, and symmetric building blocks play a vital role. The symmetric networks considered in this section are cyclic networks C_n, circulant networks $C_n(1,2)$, mobious ladder networks M_{2n}, and generalized prism networks G_m^n.

2.1. Cyclic Networks. One of the most important building blocks of complex networks is cyclic network. The vertex and edge set of a cyclic network C_n are given by $V(C_n) = \{a_i \mid 1 \leq i \leq n\}$ and $E(C_n) = \{a_i a_{i+1} \mid 1 \leq i \leq n\}$, respectively, with indices taken mod n. The network C_n is shown in Figure 1. In this section, LSRNs of cyclic network C_n are considered.

Lemma 2. *Let $a_i \in V(C_n)$, where $n \geq 3$ and $1 \leq r \leq n$. Then,*

(1) $|S\{a_r, a_{r+1}\}| = \begin{cases} n & \text{if } n \equiv 0 \pmod 2; \\ n-1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$

(2) $S\{x, y\} \in \mathscr{L}(C_n)$ *if and only if* $x = a_r$, $y = a_{r+1}$

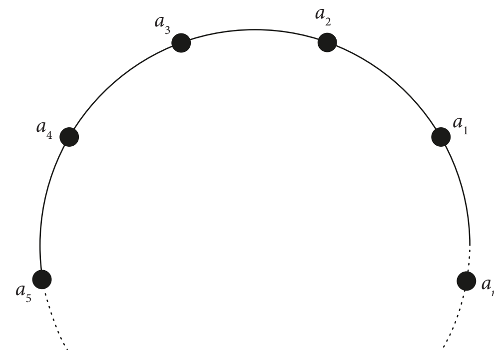

Figure 1: Cyclic network C_n.

(3) $|\cup [\mathscr{L}(C_n)]| = n$ *where* $\cup [\mathscr{L}(C_n)] = \cup_{LS(C_n) \in \mathscr{L}(C_n)} LS(C_n)$

(4) $|S\{x, y\} \cap [\cup \mathscr{L}(C_n)]| \geq \gamma(C_n)$ *for each distinct* $x, y \in V(C_n)$.

Proof. In order to prove this lemma, we proceed as follows:

(1) For $n \equiv 0 \pmod 2$, $S\{a_r, a_{r+1}\} = V(C_n)$ where as for $n \equiv 1 \pmod 2$, $S\{a_r, a_{r+1}\} = V(C_n) - \{a_{r+\lceil n/2 \rceil}\}$. Hence, $|S\{a_r, a_{r+1}\}| = n$ or $n-1$, respectively.

(2) It is clear that $S\{a_r, a_{r+1}\}$ are the only LSRNs of C_n and hence, we conclude $S\{x, y\} \in \mathscr{L}(C_n)$ if and only if $x = a_r$, $y = a_{r+1}$.

(3) From the proof of (1) and (2), we have $|\cup [\mathscr{L}(C_n)]| = n$ where $\cup [\mathscr{L}(C_n)] = \cup_{LS(C_n) \in \mathscr{L}(C_n)} LS(C_n)$.

(4) Indeed, the only pair of adjacent vertices in C_n are a_r, a_{r+1} so we have $|S\{x, y\} \cap [\cup \mathscr{L}(C_n)]| \geq \gamma(C_n)$ for each distinct $x, y \in V(C_n)$. □

2.2. Circulant Networks. The circulant network $C_n(s_1, s_2, s_3, \ldots, s_k)$ is formed by arranging the n vertices labelled a_i with the indices taken mod n cyclically and connecting each vertex a_i with k immediately following and k preceeding vertices, where $k \leq \lfloor n/2 \rfloor$. If $= \lfloor n/2 \rfloor$, then the circulant network represented by a complete graph. $C_n(1,2)$ is a circulant network with vertex set $V(C_n(1,2)) = \{a_i; 1 \leq i \leq n\}$ and edge set $E(C_n(1,2)) = \{a_i a_{i+1}, a_i a_{i+2}, a_i a_{i-1}, a_i a_{i-2}; 1 \leq i \leq n\}$. The network $C_n(1,2)$ is shown in Figure 2.

Lemma 3. *Let $a_i \in V(C_n(1,2))$, where $n \geq 6$ and $1 \leq r \leq n$. Then,*

(1) $|S\{a_r, a_{r+1}\}| = |S\{a_r, a_{r-1}\}| = 2(\lceil m + 1/2 \rceil)$, *where* $m = \lceil n - 5/4 \rceil$

(2) $S\{x, y\} \in \mathscr{L}(C_n(1,2))$ *if and only if either* $x = a_r$, $y = a_{r-1}$ *or* $x = a_r$, $y = a_{r+1}$

(3) $|\cup \mathscr{L}(C_n(1,2))| = n$ *where* $\cup \mathscr{L}(C_n(1,2)) = \cup_{LS(C_n(1,2)) \in \mathscr{L}(C_n(1,2))} LS(C_n(1,2))$

(4) $|S\{x, y\} \cap [\cup \mathscr{L}(C_n(1,2))]| \geq \gamma(C_n(1,2))$ *for each distinct* $x, y \in V(C_n(1,2))$

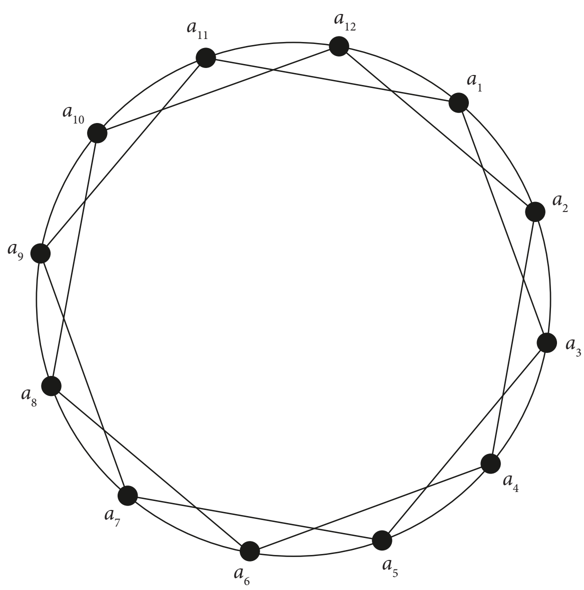

FIGURE 2: The circulant network $C_{12}(1,2)$.

Proof. The proof of this lemma is as follows:

(1) We consider the LSRNs of the vertex pair $a_r a_{r+1}$ which are $S\{a_r, a_{r+1}\} = \{a_r, a_{r+1}, \ldots, a_{r+k}, a_r, a_{r-2}, \ldots, a_{r-m}\}$ where $m = 2\lceil n - 5/4\rceil$ and $k = 2\lceil n - 5/4\rceil + 1$. Hence, due to symmetry of $C_n(1,2)$, $|S\{a_r, a_{r-1}\}| = |S\{a_r, a_{r+1}\}| = 2(\lceil m + 1/2\rceil)$.

(2) To prove this claim, we consider the LSRNs for $a_r a_{r+2}$ and $a_r a_{r-2}$. Here, following cases arise:

Case 1 $(n \equiv 0 \,(\mathrm{mod}\, 2))$
It is easy to see that the LSRNs in this case are given by $S\{a_r, a_{r+2}\} = \{a_{r+1}, a_{r+n/2+1}\}^c$. Hence, due to symmetry, $|S\{a_r, a_{r+2}\}| = |S\{a_r, a_{r-2}\}| = n - 2$.
Case 2 $(n \equiv 1 \,(\mathrm{mod}\, 2))$
This case is further subdivided into following cases:

Case 2.1 (when $n = 7 + 4k$ where $k \in Z$)
Here, we have $S\{a_r, a_{r+2}\} = \{a_{r+1}\}^c$. Hence, due to symmetry, $|S\{a_r, a_{r+2}\}| = |S\{a_r, a_{r-2}\}| = n - 1$
Case 2.2 (when $n = 9 + 4k$ where $k \in Z$)
The LSRNs in this case are given by $S\{a_r, a_{r+2}\} = \{a_{r+\lceil n/2\rceil}, a_{r+\lceil n/2\rceil+1}, a_{r+1}\}^c$. Hence, due to symmetry, $|S\{a_r, a_{r+2}\}| = |S\{a_r, a_{r-2}\}| = n - 3$.
Hence from above, we conclude $S\{x, y\} \in \mathscr{L}(C_n(1,2))$ if and only if either $x = a_r$, $y = a_{r-1}$ or $x = a_r$, $y = a_{r+1}$. Also, $|LS(C_n(1,2))| \le |S\{a_r, a_{r+2}\}|$ and $|LS(C_n(1,2))| \le |S\{a_r, a_{r-2}\}|$.

(3) From the proof of (1) and (2), we have $|(\cup_{r=1}^{n} S\{a_r, a_{r-1}\}) \cup (\cup_{r=1}^{n} Sa_r, a_{r+1})| = |\{a_i \mid 1 \le i \le n\}| = n$. Hence, $|\cup \mathscr{L}(C_n(1,2))| = n$ where $\cup \mathscr{L}(C_n(1,2)) = \cup_{LS(C_n(1,2)) \in \mathscr{L}(C_n(1,2))} LS(C_n(1,2))$.

(4) It can be concluded from the proof of (1) and (2) that $|S\{x, y\} \cap [\cup \mathscr{L}(C_n(1,2))]| \ge \gamma(C_n(1,2))$ for each distinct $x, y \in V(C_n(1,2))$. □

2.3. Mobious Ladder Network. The network obtained by introducing a twist in a prism network of order n is known as the mobious ladder network denoted by M_{2n}. It is formed by arranging its $2n$ vertices labelled a_i and b_i with the indices taken mod n cyclically and connecting each vertex a_i with b_i similar to a prism with two edges crossed. The collection of vertices and edges of mobious ladder M_{2n} is represented by $V(M_{2n}) = \{a_i, b_i; 1 \le i \le n\}$ and $E(M_{2n}) = \{a_i a_{i+1}, b_i b_{i+1}, a_j b_j, a_1 b_n, a_n b_1; 1 \le i \le n \; -1, 1 \le \; j \le n\}$, respectively. The network M_{2n} is shown in Figure 3.

Lemma 4. *Let $a_i, b_i \in V(M_{2n})$, where $n \ge 6$, $1 \le r \le n-1$ and $1 \le q \le n$. Then,*

(1) $|S\{a_r, b_{r+1}\}| = |S\{a_q, b_q\}| = |S\{a_n, b_1\}| = |S\{a_1, b_n\}| = |S\{b_r, b_{r+1}\}| = \begin{cases} 2(n-1) & \text{if } n \equiv 0 \,(\mathrm{mod}\, 2); \\ 2n & \text{if } n \equiv 1 \,(\mathrm{mod}\, 2) \end{cases}$

(2) $S\{x, y\} \in \mathscr{L}(M_{2n})$ *if and only if* $S\{x, y\} \in \{S\{a_r, b_{r+1}\}, S\{a_q, b_q\}, S\{b_r, b_{r+1}\}, S\{a_n, b_1\}, S\{a_1, b_n\}\}$

(3) $|\cup [\mathscr{L}(M_{2n})]| = 2n$ *where* $\cup [\mathscr{L}(M_{2n})] = \cup_{LS(M_{2n}) \in \mathscr{L}(M_{2n})} LS(M_{2n})$

(4) $|S\{x, y\} \cap [\cup \mathscr{L}(M_{2n})]| \ge \gamma(M_{2n})$ *for each distinct* $x, y \in V(M_{2n})$.

Proof. To prove this lemma, we proceed as follows:

(1) In order to prove this claim, we consider the following cases:

Case 1 In this specific case, when $n \equiv 0 \,(\mathrm{mod}\, 2)$, the LSRNs of the vertex pairs $a_r a_{r+1}$ and $b_r b_{r+1}$ are given by $S\{a_r, a_{r+1}\} = \{a_{r+n/2+1}, b_{r+n/2}\}^c$ and $S\{b_r, b_{r+1}\} = \{a_{r+n/2}, b_{r+n/2+1}\}^c$. For the vertex pair $a_q b_q$, the LSRN is given by $S\{a_q, b_q\} = \{a_{q+n/2}, b_{q+n/2}\}^c$ where $1 \le r \le n-1$ and $1 \le q \le n$. Also, $S\{a_1, b_n\} = \{a_{n/2+1}, b_{n/2}\}^c$ and $S\{a_n, b_1\} = \{a_{n/2}, b_{n/2+1}\}^c$. Hence, we have $|S\{a_r, b_{r+1}\}| = |S\{a_q, b_q\}| = |S\{a_n, b_1\}| = |S\{a_1, b_n\}| = |S\{b_r, b_{r+1}\}| = 2(n-1)$.
Case 2 It can be seen when $n \equiv 1 \,(\mathrm{mod}\, 2)$, all the LSRNs of M_{2n} are given by $S\{x, y\} = V(M_{2n})$ where $xy \in E(M_{2n})$. Hence, we have $|S\{a_q, b_q\}| = |S\{a_r, a_{r+1}\}| = |S\{b_r, b_{r+1}\}| = |S\{a_1, b_n\}| = |S\{a_n, b_1\}| = |V(M_{2n})|$.

(2) The only LSRNs of M_{2n} are $S\{a_r, b_{r+1}\}, S\{a_q, b_q\}, S\{b_r, b_{r+1}\}, S\{a_n, b_1\}, S\{a_1, b_n\}$, and hence, we conclude $S\{x, y\} \in \mathscr{L}(C_n)$ if and only if $S\{x, y\} \in \{S\{a_r, b_{r+1}\}, S\{a_q, b_q\}, S\{b_r, b_{r+1}\}, S\{a_n, b_1\}, S\{a_1, b_n\}\}$.

(3) From the proof of (1) and (2), we have $|\cup [\mathscr{L}(M_{2n})]| = 2n$ where $\cup [\mathscr{L}(M_{2n})] = \cup_{LS(M_{2n}) \in \mathscr{L}(M_{2n})} LS(M_{2n})$.

(4) As the only LSRNs of the pairs of adjacent vertices in M_{2n} are $\{S\{a_r, b_{r+1}\}, S\{a_q, b_q\}, S\{b_r, b_{r+1}\}, S\{a_n, b_1\}, S\{a_1, b_n\}\}$. Hence, we have $|S\{x, y\} \cap [\cup \mathscr{L}(M_{2n})]| \ge \gamma(M_{2n})$ for each distinct $x, y \in V(M_{2n})$. □

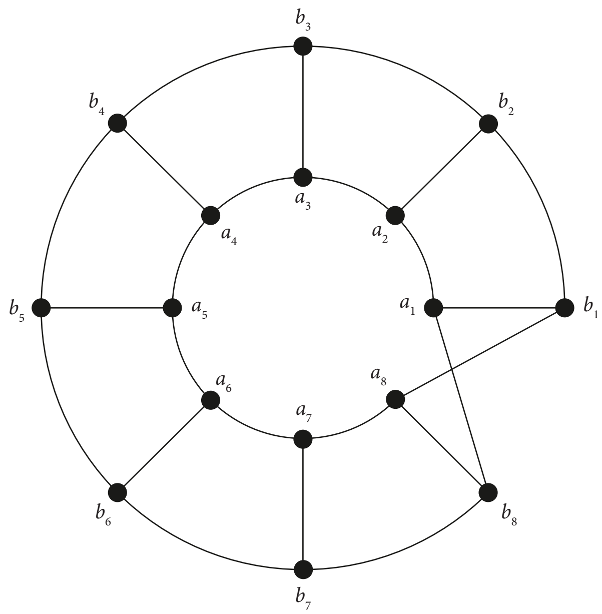

FIGURE 3: Mobious ladder network M_{16}.

2.4. Generalized Prism Network $P_m \times C_n$. Generalized prism network G_m^n is formed by the box product of networks P_m and C_n. The vertex set of G_m^n is given by $V(G_m^n) = \{a_{ij}\colon 1 \le i \le m, 1 \le j \le n\}$, and edge set is represented as $E(G_m^n) = \{\{a_{ik}a_{i(k+1)}; 1 \le i \le m, 1 \le k \le n\} \cup \{a_{st}a_{(s+1)t}; 1 \le s \le m-1, 1 \le t \le n\}\}$, respectively, where for the vertices, the first indices are taken mod m, and the second indices are taken mod n. G_m^n is shown in the Figure 4. LSRNs of generalized prism network G_m^n will be calculated in this section.

Lemma 5. *Let $a_{ij} \in V(G_m^n)$, where $n \ge 6$, $1 \le i \le m$ and $1 \le j \le n$. Then,*

(1) $|S\{a_{ij}, a_{(i+1)j}\}| = mn$ and $|S\{a_{ij}, a_{i(j+1)}\}| = \begin{cases} mn & \text{if } n \equiv 0 \pmod 2; \\ m(n-1) & \text{if } n \equiv 1 \pmod 2 \end{cases}$

(2) $S\{x, y\} \in \mathscr{L}(G_m^n)$ if and only if $x = a_{ij}$, $y = a_{i(j+1)}$ when $n \equiv 1 \pmod 2$ and $S\{x, y\} \in \mathscr{L}(G_m^n)$ if and only if $x = a_{ij}$, $y = a_{i(j+1)}$ or $x = a_{ij}$, $y = a_{(i+1)j}$ when $n \equiv 0 \pmod 2$

(3) $|\cup [\mathscr{L}(G_m^n)]| = mn$ where $\cup [\mathscr{L}(G_m^n)] = \cup_{LS(G_m^n) \in \mathscr{L}(G_m^n)} LS(G_m^n)$

(4) $|S\{x, y\} \cap [\cup \mathscr{L}(G_m^n)]| \ge \gamma(G_m^n)$ for each distinct $x, y \in V(G_m^n)$.

Proof. To prove this lemma, we proceed in the following way:

(1) It can be seen in this case when $n \equiv 0 \pmod 2$ that all the LSRNs of G_m^n are given by $S\{x, y\} = V(G_m^n)$ where $xy \in E(G_m^n)$. On the account of n being an odd number for generalized prism network G_m^n, the cardinality of LSRNs of the vertex pairs $a_{ij}a_{i(j+1)}$ is given by $|S\{a_{ij}, a_{i(j+1)}\}| = |\{a_{i(j+\lceil n/2 \rceil+1)} | 1 \le i \le m, 1 \le j \le n\}^c| = m(n-1)$. The cardinality of the LSRNs of $a_{ij}a_{(i+1)j}$ is given by $|S\{a_{ij}, a_{(i+1)j}\}| = |V(G_m^n)| = mn$.

(2) From the proof of (1), we have $S\{x, y\} \in \mathscr{L}(G_m^n)$ if and only if $x = a_{ij}$, $y = a_{i(j+1)}$ when $n \equiv 1 \pmod 2$ and $S\{x, y\} \in \mathscr{L}(G_m^n)$ if and only if $x = a_{ij}$, $y = a_{i(j+1)}$ or $x = a_{ij}$, $y = a_{(i+1)j}$ when $n \equiv 0 \pmod 2$.

(3) From (1) and (2), we note that $|\cup [\mathscr{L}(G_m^n)]| = mn$ where $\cup [\mathscr{L}(G_m^n)] = \cup_{LS(G_m^n) \in \mathscr{L}(G_m^n)} LS(G_m^n)$.

(4) From above, we conclude that

$$\left|S\{x, y\} \cap [\cup \mathscr{L}(G_m^n)]\right| \ge \gamma(G_m^n), \tag{1}$$

for each distinct $x, y \in V(G_m^n)$. □

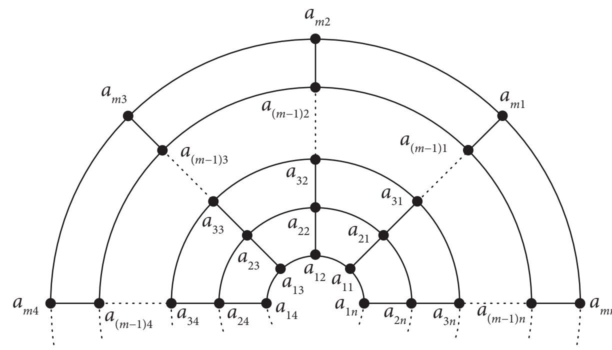

FIGURE 4: Generalized prism network $G_{m,n}$.

3. Local Fractional Strong Metric Dimension of Certain Complex Networks

In this section, LFSMD of certain complex networks is computed.

Theorem 2. *For $n \ge 3$,*

$$\operatorname{lsdim}_f(C_n) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 2; \\ \dfrac{n}{n-1}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{2}$$

Proof. To prove the above claim, we consider the following cases:

Case 1 ($n \equiv 0 \pmod 2$)

We take note of Lemma 2, $\gamma(C_n) = |V(C_n)| = n$ and $\beta(C_n) = |\cup \mathscr{L}(C_n)| = |V(C_n)| = n$. Hence, from Lemma 1, we conclude

$$\operatorname{lsdim}_f(C_n) = \sum_{s=1}^{\beta(C_n)} \frac{1}{\gamma(C_n)} = 1. \tag{3}$$

Case 2 ($n \equiv 1 \pmod 2$)

Here, from Lemma 2, $\gamma(C_n) = (n-1)$ and $\beta(C_n) = |\cup \mathscr{L}(C_n)| = n$. By using the Lemma 1, we have

$$\text{lsdim}_f(C_n) = \sum_{s=1}^{\beta(C_n)} \frac{1}{\gamma(C_n)} = \frac{n}{n-1}. \quad (4)$$

□

Theorem 3. *For* $n \geq 6$, $\text{lsdim}_f(C_n(1,2)) = n/2(\lceil m + 1/2 \rceil)$.

Proof. On account of Lemma 3, $\gamma(C_n(1,2)) = |S\{a_r, a_{r+1}\}| = |S\{a_r, a_{r-1}\}| = 2(\lceil m + 1/2 \rceil)$ where $1 \leq r \leq n$ and $m = \lceil n - 5/4 \rceil$. Moreover, $\beta(C_n(1,2)) = |\cup \mathscr{L}(C_n(1,2))| = n$. Therefore, from Lemma 1, we have

$$\text{lsdim}_f(C_n(1,2)) = \sum_{s=1}^{\beta(C_n(1,2))} \frac{1}{\gamma(C_n(1,2))} = \frac{n}{2(\lceil m + 1/2 \rceil)}. \quad (5)$$

□

Theorem 4. *For* $n \geq 6$,

$$\text{lsdim}_f(M_{2n}) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 2; \\ \dfrac{n}{n-1}, & \text{if } n \equiv 0 \pmod 2. \end{cases} \quad (6)$$

Proof. The proof of this theorem is subdivided into the following two cases:

Case 1 ($n \equiv 1 \pmod 2$)

Taking Lemma 4 into consideration, we have $\gamma(M_{2n}) = |V(M_{2n})| = 2n$ and $\beta(M_{2n}) = |\cup \mathscr{L}(M_{2n})| = |V(M_{2n})| = 2n$. Hence, from Lemma 1, the following can be concluded:

$$\text{lsdim}_f(M_{2n}) = \sum_{s=1}^{\beta(M_{2n})} \frac{1}{\gamma(M_{2n})} = 1. \quad (7)$$

Case 2 ($n \equiv 0 \pmod 2$)

In this case by considering Lemma 4, $\gamma(M_{2n}) = 2(n-1)$ and $\beta(M_{2n}) = |\cup \mathscr{L}(M_{2n})| = 2n$. Hence, from Lemma 1 we have

$$\text{lsdim}_f(M_{2n}) = \sum_{s=1}^{\beta(M_{2n})} \frac{1}{\gamma(M_{2n})} = \frac{n}{n-1}. \quad (8)$$

□

Theorem 5. *For* $n \geq 6$,

$$\text{lsdim}_f(G_m^n) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 2; \\ \dfrac{n}{n-1}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \quad (9)$$

Proof. The proof can be segregated into the following two cases:

Case 1 ($n \equiv 1 \pmod 2$)

In view of Lemma 5, $\gamma(G_m^n) = m(n-1)$ and $\beta(G_m^n) = |\cup \mathscr{L}(G_m^n)| = |V(G_m^n)| = mn$. By using Lemma 1, we have

$$\text{lsdim}_f(G_m^n) = \sum_{t=1}^{\beta(G_m^n)} \frac{1}{\gamma(G_m^n)} = \frac{n}{n-1}. \quad (10)$$

Case 2 ($n \equiv 0 \pmod 2$)

In this case using Lemma 5, $\gamma(G_m^n) = mn$ and $\beta(G_m^n) = |\cup \mathscr{L}(G_m^n)| = mn$. Hence, from Lemma 1, we conclude that

$$\text{lsdim}_f(G_m^n) = \sum_{s=1}^{\beta(G_m^n)} \frac{1}{\gamma(G_m^n)} = 1. \quad (11)$$

□

4. Application

In this section, an application of LFSMD is considered in the information processing and co-ordination of large-scale interconnection networks. Complex large-scale interconnection networks used in the design of local area networks, distributed computer systems, and telecommunication networks have been constructed based on VLSI circuit technology. In telecommunication networks, many stations are placed at short distances to share data at a very high speed, and the main objective is to optimize the exchange of data with an efficient network topology. For an illustrative case, consider a telecommunication network consisting of different stations placed at nodes of a network $C_6(1,2)$ as shown in Figure 5. In order to maintain connectivity, certain stations are required to maintain their working capacity at an optimal level. These stations are required to be at a uniform distance from all stations in order to achieve optimal connectivity. The nodes of the network $C_6(1,2)$ are x_1, x_2, x_3, x_4, x_5, x_6. The LSRNs of $C_6(1,2)$ are given as follows: $S\{x_1, x_2\} = S\{x_2, x_4\} = S\{x_4, x_5\} = S\{x_1, x_5\} = \{x_1, x_2, x_4, x_5\}$, $S\{x_1, x_3\} = S\{x_3, x_4\} = S\{x_4, x_6\} = S\{x_1, x_6\} = \{x_1, x_3, x_4, x_6\}$, $S\{x_2, x_3\} = S\{x_3, x_5\} = S\{x_5, x_6\} = S\{x_2, x_6\} = \{x_2, x_3, x_5, x_6\}$. For any given network, LSRN is the collection of nodes that are at unequal distances from a pair of adjacent nodes, and therefore, by assigning minimum weights to the nodes from LSRNs of the network, there will be minimum reliance on these nodes, and an optimal exchange of data is achieved in certain complex large-scale networks. In a network, stations are placed in such a way that the distance of every node of the network to the station is minimum which aids in the sharing of data at a very high speed. Taking Lemma 1 into consideration, if weight of 1/4 is assigned to all the nodes in the union of all LSRNs with minimum cardinality and zero to the remaining vertices of $C_6(1,2)$, then optimal exchange of data is achieved.

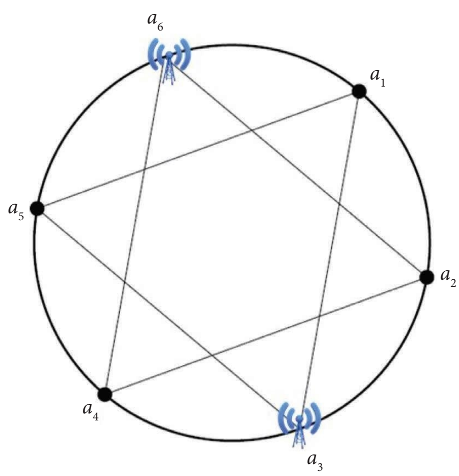

Figure 5: Telecommunication network using circulant network $C_6(1,2)$.

5. Conclusion

In this paper, LFSMD of complex networks is computed with the building blocks of complex networks considered as the symmetric networks such as cyclic networks C_n, circulant networks $C_n(1,2)$, mobious ladder networks M_{2n}, and generalized prism networks G_m^n.

Problem 1. Compute the LFSMD of some general classes of convex polytopes.

Data Availability

All the data used to support the findings of this study are included within this article and are available from corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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