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On the Zero Divisor and Cayley Graphs of Some Classes of the 2-Radical Index of Nilpotence Finite Local Rings

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> **Original Research Article**

Abstract

The characterization of finite local rings via the well known structures of their zero divisor graphs and cayley graphs remains an open problem. Some classes of completely primary finite rings which are local, have however been characterized by the compartments of their units and zero divisors where the classification of the unit groups have been done using the Fundamental Theorem of finitely generated Abelian Groups while the zero divisors have been characterized via the zero divisor graphs. This paper characterizes the zero divisor graphs $\Gamma(R)$ and cayley graphs $CAY(R)$ where R is a finite local ring with 2-radical index of Nilpotence. These two classes of graphs have been completely described and compared using their algebraic properties. Some of the graphs have been drawn for purposes of their comparison. The methods of study involved partitioning the ring under consideration into mutually disjoint subsets of invertible elements and zero divisors and determining their graphs using case by case basis discovery of their structural properties. Some symmetric groups associated with the graphs studied have also been given.

Keywords: Zero divisor graphs; Cayley graphs; 2-radical index of nilpotence finite local rings.

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1   Introduction

The definition of terms and notations used in this paper, relating to Graph Theory can be attributed to Diestel[1]. Other useful terminologies and studies related to Graph Theory and Ring Theory can be obtained from the remaining references [2-16].

Let R be a commutative ring with unity and let $Z(R)$ be the set of zero divisors. The study of R in which the subset of zero divisors forms a unique maximal ideal has been extensively done yielding interesting results (see[2], [3], [4] and [5]). Some attempts have also been made to classify finite commutative rings using the structures of their zero divisor graphs. For instance, Beck[6], Anderson and Naseer[7], Anderson and Badawi[8], Anderson and Livingston[9] among others have determined various graph invariants and characteristics associated with the Total graphs and the the zero divisor graphs of commutative rings. These studies are however nonconclusive. On the other hand, the classification of finite local rings R using the Cayley graphs $CAY(R)$ of R is still very scanty in the existing Literature. The study of the Cayley graphs was introduced by Authur Cayley in 1878 to explain the concept of abstract groups which are described by a set of generators. They are the undirected graphs whose vertices are elements of R and such that two distinct vertices x and y are adjacent if and only if $x - y \in Z(R)$, so they are akin to $\Gamma(R)$. Cayley graphs are important in relating Group Theory and Graph Theory. Therefore the relationship between the zero divisor graphs $\Gamma(R)$ and the cayley graphs $CAY(R)$ of finite local rings is an important dimension of classification worth advancing. In this paper, vital graph algebraic parameters such as diameter, girth, colouring, binding number, connectivity among others, of zero divisor and the Cayley graphs of R have been characterized. We have also obtained the symmetric groups associated with some of these graphs.

Let $R_\circ = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and characteristic p^k such that p, k, r are invariants and U an h-generated $R_\circ$-module. We provide the construction of $R = R_\circ \oplus U$ in which the set of zero divisors $Z(R)$ satisfies the condition $(Z(R))^2 = 0$.

We follow Raghavendran's Principle as applied in the construction of Completely Primary finite Rings.

2   Construction I : $2-$ Radical Index of Nilpotence Finite Local Rings of characteristic p

For every prime integer p and a positive integer r, let $R_o = GR(p^r, p)$. Now for all $i\epsilon[1, \cdots, h]$, let $u_i\epsilon Z(R)$ and U be an h-dimensional R_o-module generated by $\{u_1, u_2, \cdots, u_h\}$ so that $R = R_o \oplus U$ is an additive Abelian Group. On R, define multiplication as follows:

$$(x_o, x_1, \cdots, x_h)(y_o, y_1, \cdots, y_h) = (x_oy_o, x_oy_1 + x_1y_o, \cdots, x_oy_h + x_hy_o).$$

It is well known that the multiplication turns R into a commutative ring with identity $(1, 0, \cdots, 0)$ (see[10]). Moreover,

$$Z(R) = R_o \oplus R_ou_1 \oplus \cdots R_ou_h$$

and

$$(Z(R))^2 = (0).$$

Given R to be the ring in construction I above, the structural and algebraic properties of such a ring can be deduced intuitively from the following propositions:

Proposition 1. *(The Graph* $\Gamma(R)$

Let R be the Local ring described by construction I above and let $Z(R)$ be the set of its zero divisors and R^*, the set of units. Then, the following are the general properties of the zero-divisor graph $\Gamma(R)$ of R;

(i) $\mid V(\Gamma(R)) \mid = p^{rh} - 1$.

(ii) $\Gamma(R)$ *is complete.*

(iii) $\Gamma(R) \cong K_{p^{rh}-1}$.

(iv) $Diam(\Gamma(R)) = 1$.

(v) $Girth(\Gamma(R)) = \begin{cases} \infty, & h\text{=1, p=2,3;} \\ 3, & elsewhere \end{cases}$

(vi) $b(\Gamma(R)) \leq \infty$.

(vii) $\Delta(\Gamma(R)) = p^{rh} - 1$.

(viii) $\chi(\Gamma(R)) \leq p^{rh} - 2$.

Proof.   (i) From construction I, $R = R_\circ \oplus U$ where $Dim_{R_\circ} U = h$. So $R = R_\circ \oplus R_\circ u_1 \oplus, \cdots, \oplus R_\circ u_h$. This implies that $\mid R \mid = p^{(h+1)r}$ $\forall h, r \epsilon \mathbb{Z}^+$. Now, $Z(R) = R_\circ u_1 \oplus R_\circ u_2 \oplus, \cdots, \oplus R_\circ u_h$. $\Rightarrow \mid Z(R) \mid = p^{hr}$.

But the vertices of $\Gamma(R)$ are all the elements of $Z(R) \setminus \{0\}$

Thus $V(\Gamma(R)) = (Z(R))^\star = \{x \in Z(R) \setminus \{0\}\}$

$$\Rightarrow \mid (Z(R))^* \mid = \mid V(\Gamma(R)) \mid$$

But

$$\mid (Z(R))^* \mid = \mid Z(R) \setminus \{0\} \mid$$
$$= p^{hr} - 1$$
$$\therefore \mid V(\Gamma(R)) \mid = p^{hr} - 1$$

as required.

(ii) Let $(0, x_1, \cdots, x_h)(0, y_1, \cdots, y_h) \epsilon Z(R)^\star$. Then using construction I, the product of the pair of the non-zero zero divisors is

$$(0, x_1, x_2, \cdots, x_h)(0, y_1, y_2, \cdots, y_h) = (0, 0, \cdots, 0)$$

and every other pair of the divisors are members of $Ann(Z(R))^\star$. So the set $(Z(R))^* = Ann(Z(R) \setminus \{0\})$ and thus members of $Z(R) \setminus \{0\}$ are pairwise adjacent implying that the graph $\Gamma(R)$ whose vertices are $Z(R) \setminus \{0\}$ is complete.

(iii) From (ii), $\Gamma(R)$ is complete and $\mid V(\Gamma(R)) \mid = p^{hr} - 1$. So the graph is denoted $K_{\mid V(\Gamma(R)) \mid} = K_{p^{hr}-1}$. On the other hand $\Gamma(R)$ is $p^{hr} - 1$-partite since the total number of independent set of vertices is $p^{hr} - 1$. Thus $\Gamma(R) \cong K_{p^{hr}-1}$.

(iv)$Diam(\Gamma(R)) = sup\{d(x,y)|x, y \in Z(R) \setminus 0\}$. Since $(Z(R))^* = Z(R) - \{0\}$, and for all distinct $x, y \in Z(R) - \{0\}$, $xy = 0$, we have that $d(x, y) = 1$. So, $sup\{d(x,y)\} = 1 \; \forall x, y \in V(\Gamma(R))$. This implies that

$$diam(\Gamma(R)) = 1.$$

(v) A complete graph $\Gamma(R)$ on n vertices is denoted that K_n. When $r = 1$, $h = 1$ and $p = 2, 3$, $n = (p^{rh} - 1) \leq 2$. So girth $\Gamma(R) = \infty$. Otherwise,$\forall \; n = (p^{rh} - 1) > 2$, it is well known from Anderson and Livingston[9] that

$$girth(\Gamma(R)) \leq 2diam(\Gamma(R)) + 1.$$

Since $diam(\Gamma(R)) = 1$, the result readily follows.

(vi) Let S be the set of vertices of minimal degree. By definition, $b(\Gamma(R)) = \frac{|N(S)|}{|S|}$ taken over all $\Phi \neq S \subseteq V(\Gamma(R))$ such that $V(\Gamma(R)) = N(S)$. Clearly $S = \Phi$, so that $\mid S \mid = 0$ thus $b(\Gamma(R)) = \infty$. Therefore, the vertices of $\Gamma(R)$ are well bound together and so the edges are fairly distributed.

(vii) $\mid V(\Gamma(R)) \mid = p^{rh} - 1$. So let $u_i \in \Gamma(R)$ be a vertex such that $u_i.u_j = 0$; $\forall \; i, j$, thus each vertex is adjacent to every other vertex except itself for avoidance of loops. Therefore, the number of vertices adjacent to u_i is $(p^{rh} - 1) - 1$. So $\Delta(\Gamma(R)) = p^{rh} - 2$.

(viii) The minimum number of colors that can be assigned to each vertex relates to the maximum degree of each vertex in $\Gamma(R)$. Therefore it can be established that $\chi(\Gamma(R)) \leq p^{rh} - 2$.

$\square$

Remark 1. In constructing the Cayley graph $CAY(R)$, we consider all elements of R as vertices and take any two members of R to be adjacent if and only if $x - y \in Z(R)$. Moreover, we denote the cayley graph $CAY(R)$ by G in the sequel.

Proposition 2. *(**The Graph** G) Let R be the Local ring described by construction I above and let $Z(R)$ be the set of its zero divisors. Then the Cayley graph G will have p copies of complete subgraphs. Moreover,*

(i) $\mid V(G) \mid = p^{1+hr}$.

(ii) G is disconnected hence incomplete.

(iii) G is isomorphic to p copies of $K_{p^{hr}}$

(iv) $Diam(G) \leq \infty$.

(v) $Gr(G) \leq \infty$

(vi) $b(G) \leq 1$

(vii) $\Delta(G) = p^{hr} - 1$.

(viii) $\chi(G)) \leq p^{hr}$.

(ix) $\omega(G) \leq p^{hr}$

(x) $C_H(G) = 0$

Proof. (i) From construction I, we have that $R_\circ = GR(p^r, p)$ and $R = R_\circ \oplus U$. So U is an $R_\circ$-module generated by $\{u_1, \cdots, u_h\}$ elements where $h = 2$. It is clear that $Z(R) = R_\circ u_1 \oplus R_\circ u_2 \oplus_\circ u_h$ and every element in G is of the form $(r_\circ, r_1, r_2, \cdots, r_h)$.
Since

$$\mid R \mid = \mid R_\circ \mid \mid U \mid = p^{1+hr},$$

then $\mid V(G) \mid = p^{1+hr}$ which establishes (i).

(ii) Note that in Cayley graphs every element of R is a vertex and for any two elements $x, y \epsilon R$ to be adjacent,$x - y \epsilon (Z(R))$. Thus G has p complete disconnected subgraphs and hence, is incomplete.

(iii) This is clear Since by (ii), G is not complete. However we have p complete subgraphs of G, namely, $G_1 \cong K_{p^{hr}}$, $G_2 \cong K_{p^{hr}} \cdots G_n \cong K_{p^{hr}}$ This implies that $G \cong p$ copies of $k_{p^{hr}}$

(iv) $Diam(G) = sup\{d(x, y) | x, y \in R\}$ and that for all distinct $x, y \in R$, $x - y \in (Z(R))$. Now, G is having p complete disconnected subgraphs and so we have that $d(x, y) = 1$. for every subgraph. However, since there is no connection between the p subgraphs, the $sup\{d(x, y)\} = \infty \; \forall x, y \in G$. This implies that $diam(G) \leq \infty$.

(v) A complete graph G with n vertices is denoted that K_n. Now since G is having p copies of $K_{p^{rh}}$, it is clear that G has no cycle and so the girth is ∞.

(vi) Let S be the set of vertices of minimal degree. By definition, $b(G) = \frac{|N(S)|}{|S|}$ taken over all $\Phi \neq S \subseteq V(G)$ such that $V(G) = N(S)$. In fact $S = N(S) \setminus Ann(Z(R))$. However the $Ann(Z(R)) = 0$, thus $| N(S) |=| S |$, hence the $b(G) = 1$.

(vii) $| V(G) |= p^{1+hr}$ and that G is not a complete graph. So let $u_i, u_j \in Gi, j = 1, 2, \cdots$ such that $u_i \neq u_j$ be a vertex such that $u_i - u_j \in Z(R)$, we see that each vertex in each subgraph is adjacent to each other except itself for avoidance of loops. Therefore the number of vertices adjacent to u_i in each subgraph is $p^{hr} - 1$. Implying that $\Delta(G) = p^{hr} - 1$.

(viii) Since $\Delta(G) = p^{hr} - 1$ and each of the subgraphs of G is complete, then the minimum number of colors that can be assigned to each vertex is equivalent to the maximum degree of each vertex in G plus one since the vertex itself is also assigned a colour. Therefore it can be established that $\chi(G) = p^{rh}$.

(ix)The clique number of G is the order of the largest complete subgraph of G. Now, the subgraphs G_1 ,G_2 $\cdots$ $G_n$are the largest and of order p^{hr}, it then follows that $\omega(G) \leq p^{hr}$.

(x)By definition, let $G = (V, E)$ be a nontrivial graph of order m and let $u \in V$ be a vertex of G. Then the Harmonic centrality of u is defined by;

$$H_G(u) = \frac{R_G(u)}{m - 1}$$

where $R_G(u) = \sum_{x \neq u} \frac{1}{d(u,x)} : x \in V \setminus u$
and $d(u, x)$ is the shortest distance between $u, x \in G$ with $\frac{1}{d(u,x)} = 0$ if there is no path between u and x.

From the Harmonic Centrality of u, we can define the Harmonic Centralization of G of order m as;

$$C_H(G) = \frac{\sum_{i=1}^{m}\{H_{G_{max}}(u) - H_G(u_i)\}}{\frac{m-2}{m}}$$

where $H_{G_{max}}(u)$ is the largest Harmonic centrality of u.

Now from the graph G,$m =| V |$. We have that all the vertices have equal harmonic centrality and so the harmonic centralization of every vertex will be 0

$\square$

Remark 2. : In order to characterize the classes of Rings considered in this paper, completely we must describe both the unit compartments R^* and the zero divisor $Z(R)$ compartments completely.

Example 1. *Consider the ring* $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. *From the consruction I, we see that* $p = 2$, $h = 2$, $k = 1$, $r = 1$.

Then

$$R = \{0, 1\} \oplus \{0, 1\} \oplus \{0, 1\}$$
$$R = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \oplus \{(0, 1)\}$$
$$R = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$

The zero divisors of R are of the form

$$Z(R) = \{(0, r, s) : r, s \in R_0\}$$

Thus from the ring R,

$$Z(R) = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1)\}$$

and the set or non-zero zero divisors are

$$Z(R) \setminus \{0\} = \{(0, 0, 1), (0, 1, 0), (0, 1, 1)\}$$

Now, the order of the ring R is given by; $\mid R \mid = 8, = 2^{3r} = 2^3 = p^{(h+1)r}$ since $h = 2$ and $r = 1$. Thus, the order of the zero divisors of R is;

$$\mid Z(R) \mid = 4 = 2^2 = p^{hr}.$$

and the order of the non-zero zero divisors is given by;

$$Z(R) \setminus \{0\} = p^{hr} - 1.$$

Moreover, the set of the units of this ring is of the form;

$$R^* = 1 + Z(R)$$

$$= \{(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$

and the order of the units is given by;

$$\mid R^* \mid = 4$$
$$R^\star = \mathbb{Z}_{p^r - 1} \times 1 + (Z(R))$$

Thus the general structure of the units of R is

$$R^* \cong 1 + Z(R)$$

$$= 1 + r_1 u_1 + r_2 u_2 \mid r_i \epsilon \mathbb{Z}_p, u_i \epsilon Z(R) : 1 \leq i \leq 2$$
$$= (1, r_1, r_2)$$

We can get the generators of the normal subgroups of $1 + Z(R)$ in $R^\star$ as follows;

$$(1, 0, 0)(1, 0, 0) = (1, 0, 0)$$

which is an identity.

$$o(1, 0, 0) = 2$$

so the cyclic group generated by the element $(1, 0, 0)$ is isomorphic to $\mathbb{Z}_2$, and we call it G_1 i.e

$$< (1, 0, 0) > \cong \mathbb{Z}_2 = G_1.$$

In a similar manner,

$$o(1, 0, 1) = 2$$

and

$$< (1, 0, 1) > \cong \mathbb{Z}_2 = G_2,$$
$$o(1, 1, 0) = 2$$

and

$$< (1, 1, 0) > \cong \mathbb{Z}_2 = G_3,$$
$$o(1, 1, 1) = 2$$

and

$$< (1, 1, 1) > \cong \mathbb{Z}_2 = G_4.$$

Therefore,

$$G_1 = \{(1, 0, 0)(1, 0, 0)\}$$
$$G_2 = \{(1, 0, 1)(1, 0, 0)\}$$
$$G_3 = \{(1, 1, 0)(1, 0, 0)\}$$
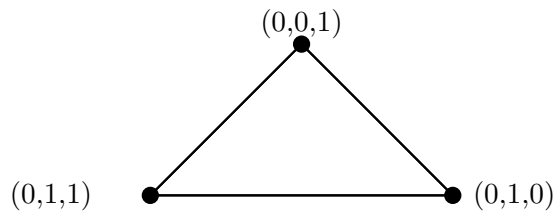$$G_4 = \{(1, 1, 1)(1, 0, 0)\}$$

We notice that;

(i) G_1, G_2, G_3 and G_4 are normal subgroups of $Z(R)$

(ii) $G_1 \cap G_2 \cap G_3 \cap G_4 = (1, 0, 0)$ identity.

From the illustration above, it follows that the set of all the non-zero zero divisors in R is given by;

$$(Z(R)) \setminus \{0\} = \{(0, 0, 1), (0, 1, 0), (0, 1, 1)\}.$$

Thus the graph $\Gamma(R)$ of the ring illustrated above is as shown below;

Thus, from proposition 1 above, $\Gamma(R)$ has the following algebraic properties:

The $\mid V(\Gamma(R)) \mid = 3$, $\Gamma(R)$ is complete, $\Gamma(R) = K_3$, $diam(\Gamma(R)) = 1$, $girth(\Gamma(R)) = 3$, $b(\Gamma(R)) = \infty$, $\Delta(\Gamma(R)) = 2$, $\chi(\Gamma(R)) \leq 2$, the number of cliques is 1 and the $\omega(\Gamma(R)) = 3$ (clique number).

Below is the Cayley graph $CAY(R)$ denoted by G, of the same ring above :

Example 2. *Consider the ring R of construction I such that $p = 2$, $h = 2$, $k = 1$ and $r = 1$ so that $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*
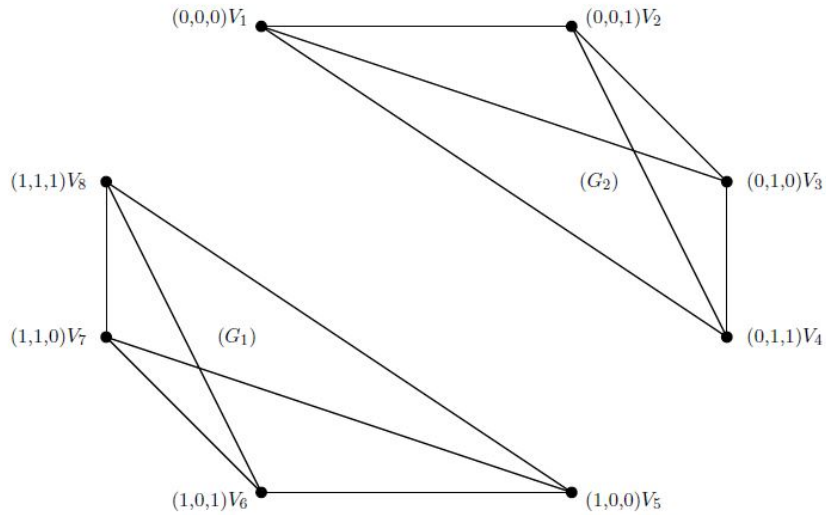
Then

$$R = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \oplus \{0, 1\}$$
$$R = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$

Thus;

$$CAY(R) = \{(x, y) \mid x, y \in R, x - y \in Z(R)\}$$
$$R = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$
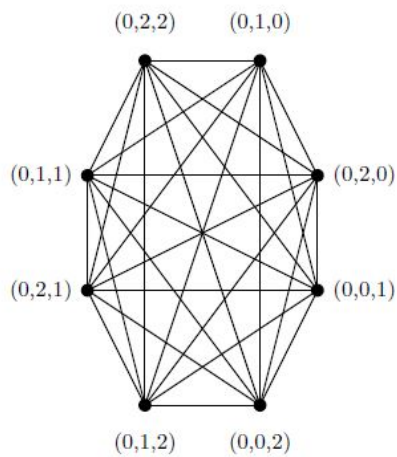
The cayley graph $CAY(R)$ is shown below

Now, G has the following properties:

The $\mid V(G) \mid = 8$, $diam(G) = \infty$, $girth(G) = \infty$, $b(G) = 1$, $\Delta(G) = 3$, $\chi(G)4$, $\omega(G) = 4$ and the $C_H(G) = 0$

Example 3. *Consider the ring R of construction I such that $p = 3$, $h = 2$, $k = 1$ and $r = 1$ so that $R = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ then, the set of non zero-zero divisors of this ring is:*

$$(Z(R)) \setminus \{0\} = \{(0,1,0),(0,2,0),(0,0,1),(0,0,2),(0,1,2),(0,2,1),(0,1,1),(0,2,2)\}.$$

Therefore, $\Gamma(R)$ of this ring is represented by;

$\Gamma(R)$ *has the following properties:*

The $| V(\Gamma(R)) | = 8$, $\Gamma(R)$ *is complete,* $\Gamma(R) = K_8$, $diam(\Gamma(R)) = 1$, $girth(\Gamma(R)) = 3$, $b(\Gamma(R)) = \infty$, $\Delta(\Gamma(R)) = 7$, $\chi(\Gamma(R)) \leq 7$ *and* $\omega(\Gamma(R)) = 8$.

We provide the graph (G) of the same ring as follows:

Example 4. *Consider the ring R of construction I such that $p = 3$, $h = 2$, $k = 1$ and $r = 1$ so that*

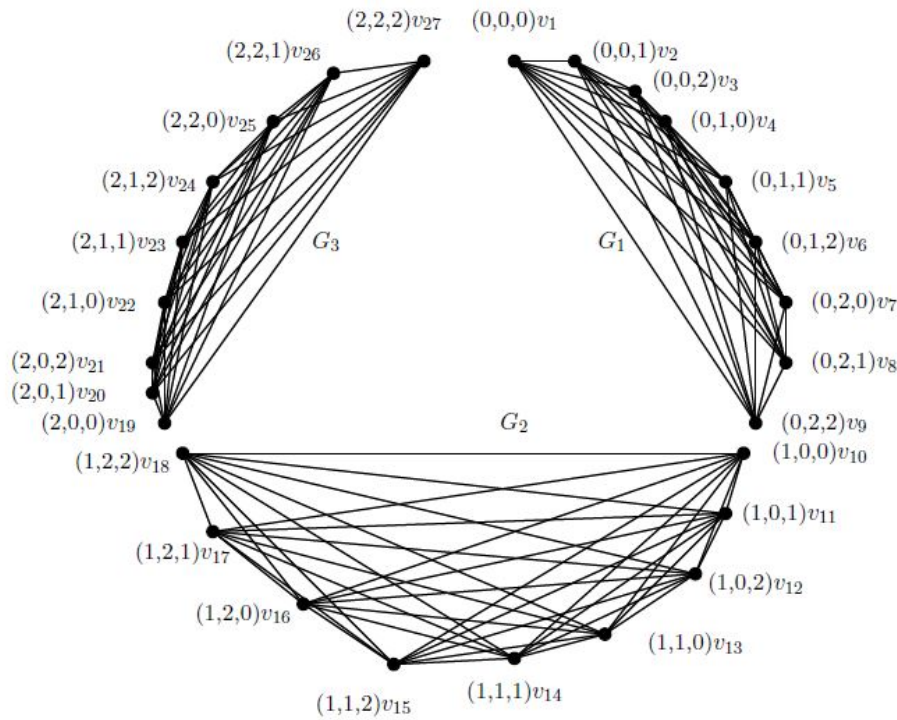$$R = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$$

so that

$$R = \{(0,1,2)(0,1,2)(0,1,2)\}$$
$$R = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\} \oplus \{0,1,2\}.$$
$$R = \{(0,0,0),(0,0,1),(0,0,2),(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2),(1,0,0),$$
$$(1,0,1),(1,0,2),(1,1,0),(1,1,1),(1,1,2),(1,2,0),(1,2,1),(1,2,2),(2,0,0),$$
$$(2,0,1),(2,0,2),(2,1,0),(2,1,1),(2,1,2),(2,2,0),(2,2,1),(2,2,2)\}$$

Therefore, G of this ring is represented by;

G has the following properties:

The $| V(G) | = 27$, $diam(G) = \infty$, $girth(G) = \infty$, $b(G) = 1$, $\Delta(G) = 8$, $\chi(G) = 9$, $\omega(G) = 9$, *and* $C_H(G) = 0$.

From the cases studied above, we provide the following two propositions on the order of edges and vertices of $\Gamma(R)$ of the ring under construction I.

Proposition 3. *Let R be a characteristic p local ring of the construction I. Then given $E(\Gamma(R))$,*

$$|E(\Gamma(R))| = \frac{1}{2}(p^{hr} - 1)(p^{hr} - 2)$$

and

$$\sum_{i=1}^{hr-1} \Delta(V_i) = (p^{hr} - 1)(p^{hr} - 2)$$

Proof. We deduce inductively from the examples given that $\Gamma(R)$ is complete on $p^{hr} - 1$ vertices. Now each of the $p^{hr} - 1$ vertices of $\Gamma(R)$ can be labeled as $1, 2, \cdots, p^{hr} - 1$. Thus

$$V_1 \longleftrightarrow V_2, \cdots, V_{(p^{hr}-2)}, V_2 \longleftrightarrow \cdots V_{(p^{hr}-3)}$$

and generally the $(n-1)^{th}$ vertex will be adjacent to $(p^{hr} - n)$ vertices. Therefore the sum of the edges of $\Gamma(R)$ is given by:

$$\sum_{i=1}^{n} E_i = (p^{hr} - 2) + (p^{hr} - 3) + (p^{hr} - 4) + \cdots + 2 + 1$$

$$= \frac{1}{2}(p^{hr} - 1)(p^{hr} - 2)$$

$$\therefore |E(\Gamma(R))| = \frac{1}{2}(p^{hr} - 1)(p^{hr} - 2)$$

Next, since $\Gamma(R)$ is complete, every $V_i \in \Gamma(R)$, $\therefore i = 1, \cdots, (p^{hr} - 1)$ is adjacent to each other. So each edge E_i is incident to 2 vertices V_i and V_j, $i \neq j$. Therefore the number of degrees will be double so that

$$\sum_{i=1}^{hr-1} \Delta(V_i) = 2\{\frac{1}{2}(p^{hr} - 1)(p^{hr} - 2)\}$$

$$= (p^{hr} - 1)(p^{hr} - 2)$$

as required. $\qquad\square$

Proposition 4. *Let R be the Local ring of construction I so that the charR=p and let $Z(R)\setminus\{0\} = p^{hr} - 1$ as required. Then:*

$$|V(\Gamma(R))| = \frac{1}{p^{hr} - 2}[\sum_{i=1}^{p^{hr}-1} \Delta(V_i)]$$

or

$$|V(\Gamma(R))| = \frac{2|E|}{p^{hr} - 2}$$

Proof. We need to show that the degrees of the vertices and edges are related. Now,

$$|Z(R) \setminus \{0\}| = |V| = p^{hr} - 1$$

and

$$\sum_{v \epsilon V} deg(V) = (p^{hr} - 1)(p^{hr} - 2)$$

Implies that

$$\frac{|V(\Gamma(R))|}{\sum_{v \epsilon V} deg(V)}$$

$$= \frac{p^{hr} - 1}{(p^{hr} - 1)(p^{hr} - 2)}$$
$$= (p^{hr} - 2)^{-1}$$

On the other hand $|E| = \frac{1}{2}(p^{hr} - 1)(p^{hr} - 2)$ and

$$\frac{2|E|}{p^{hr} - 2} = \frac{(p^{hr} - 1)(p^{hr} - 2)}{(p^{hr} - 2)} = (p^{hr} - 1) = |V(\Gamma(R))|$$

Thus the result follows for both cases. $\qquad\square$

3   Construction II: $2-$ Radical Index of Nilpotence Finite Local Rings of Characteristic p^2

For any prime integer p and a positive integer r, let $R_\circ = GR(p^{2r}, p^2)$ and $K = R_\circ/pR_\circ$ so that $U = K^h$ is an $R_\circ$-module generated by $\{u_1, u_2, \cdots, u_h\}$, where $pu_i = 0, \forall i = 1 - h$ and $u_i \in R_\circ$. On the additive group $R = R_\circ \oplus K^h$, define multiplication as follows;

$$(x_\circ, \overline{x}_1, \cdots, \overline{x}_h)(y_\circ, \overline{y}_1, \cdots, \overline{y}_h) = (x_\circ y_\circ, x_\circ \overline{y}_1 + \overline{x}_1 y_\circ, \cdots, x_\circ \overline{y}_h + \overline{x}_h y_\circ).$$

It is well known that the multiplication defined above turns $R = R_\circ \oplus K^h$ into a commutative local ring with identity $(1, \overline{0}, \cdots, \overline{0})$. Indeed $Z(R)$ satisfies:

(i) $Z(R) = pR_\circ \oplus K^h$.

(ii) $(Z(R))^2 = 0$. Where $K^h = \{r_\circ u_1 + r_\circ u_2 + \cdots + r_\circ u_h\}, u_i \epsilon (Z(R))$

Proposition 5. *(**The graph** $\Gamma(R)$)*

Let R be a local ring of construction II. Then:

(i) $| V(\Gamma(R)) | = p^{1+h} - 1$.

(ii) $Diam(\Gamma(R)) = 1$.

(iii) $Girth(\Gamma(R)) = \begin{cases} \infty, & h\text{=}1,\ p\text{=}2; \\ 3, & elsewhere \end{cases}$

(vi) $b(\Gamma(R)) = \frac{p^2 - 1}{(p^{1+h} - 1) - (p^2 - 1)}$

Proof.   (i)$Z(R) = R_\circ u_1 \oplus \cdots \oplus R_\circ u_h$ is a maximal ideal of R. Thus the quotient $R/Z(R)$ is a field of order p.

Now consider an element $\alpha \in R/Z(R)$ but α is neither $\overline{0}$ nor $\overline{1}$ then $(R/Z(R)) \setminus \{0\} = < a >$ and the order of a is

$\circ(a) = p - 1$. Therefore each element that lies outside $Z(R)$ has an inverse. Thus

$$| Z(R)^* | = p^{1+h} - 1$$
$$\Rightarrow | \Gamma(R) | = p^{1+h} - 1$$

(ii)Then $Ann(Z(R)) = pR_\circ$. Now for some $u \in pR_\circ$ there exists some $m \in Z(R) \setminus \{0\}$ such that $u, m \in pR_\circ$. But $u, m, w = 0$ where $w \in Ann(Z(R)) = pR_\circ$.

Thus $diam(\Gamma(R)) = 1$.

(iii) If $r = 1$, $p = 2$, $h = 1$ then $Z(R) \setminus \{0\} = \{(0,1), (2,0), (2,1)\}$. So $(2,0)$ is adjacent to the other two vertices. Moreover, any $y \in Z(R) \setminus \{0\}$ is adjacent to $x \in Ann(Z(R) \setminus \{0\})$ since $| Ann(Z(R) \setminus \{0\}) | = p - 1$, the result follows.

(iv) Let $N(S) = Ann(Z(R)) = pR_\circ$ and $S = V(\Gamma(R)) \setminus N(S)$. Sine $\mid N(S) \mid = p^2 - 1$ and $\mid S \mid = (p^{h+1} - 1) - (p^2 - 1)$

$$\therefore b(\Gamma(R)) = \frac{N(S)}{\mid s \mid} = \frac{p^2 - 1}{(p^{h+1} - 1) - (p^2 - 1)}.$$

$\square$

Proposition 6. *If $r = 1$, $k = 2$ and $h = 1$. Then $\Gamma(R)$ is p-partite.*

Proof. This involves partitioning $\Gamma(R)$ into disjoint subsets;
Let
$$V_{ip} = \{ip\} : 1 \leq i \leq p - 1$$
$$V_1 = Z(R) - Ann(Z(R))$$
Then
$$V(\Gamma(R)) = \bigcup_{i=1}^{p-1} (V_{ip} \cup V_1)$$

and each of the pairs of the subsets are disjoint.

$\square$

We provide some cases based on the propositions above;

Example 5. *Consider the ring R of construction II such that $p = 2$, $h = 1$, $k = 2$ and $r = 1$ so that $R = \mathbb{Z}_4 \oplus \mathbb{Z}_2$ then,*
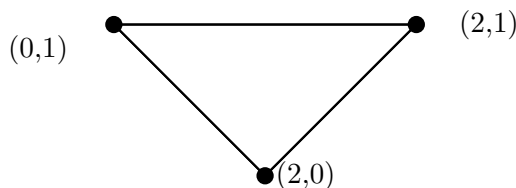$$R = \{(0, 1, 2, 3)\} \oplus \{0, 1\}.$$

$$R = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)\}.$$

$$Z(R) = \{(0, 0), (0, 1), (2, 0), (2, 1)\}.$$

$$Z(R) \setminus \{0\} = \{(0, 1), (2, 0), (2, 1)\}.$$

Therefore $\Gamma(R)$ with the vertices $(Z(R))^*$ is as shown below;
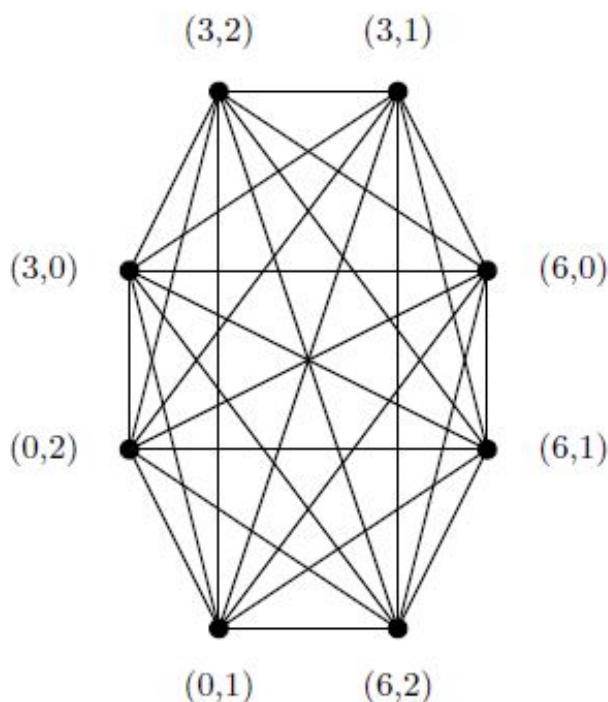

$\Gamma(R)$ *has the following algebraic properties:*

The $\mid V(\Gamma(R)) \mid = 3$, $\Gamma(R)$ is complete, $\Gamma(R) = K_3$, $diam(\Gamma(R)) = 1$, $girth(\Gamma(R)) = 3$, $b(\Gamma(R)) = \infty$, $\Delta(\Gamma(R)) = 2$, $\chi(\Gamma(R)) \leq 2$, $\omega(\Gamma(R)) = 3$ and $\Gamma(R)$ is a planar graph.

Example 6. *Consider the ring R of construction II such that $p = 3$, $h = 1$, $k = 2$ and $r = 1$ so that $R = \mathbb{Z}_9 \oplus \mathbb{Z}_3$ then, the set of non zero-zero divisors of this ring is:*

$$Z(R) \setminus \{0\} = \{(0, 1), (0, 2), (3, 0), (3, 1), (3, 2), (6, 0), (6, 1), (6, 2)\}.$$

Therefore, $\Gamma(R)$ of this ring is represented by;

$\Gamma(R)$ *has the following properties:*
The $| V(\Gamma(R)) |= 8$, $\Gamma(R)$ *is complete,* $\Gamma(R) = K_8$, $diam(\Gamma(R)) = 1$, $girth(\Gamma(R)) = 3$, $b(\Gamma(R)) = \infty$,
$\Delta(\Gamma(R)) = 7$, $\chi(\Gamma(R)) \le 7$, $\omega(\Gamma(R)) = 8$, $\Gamma(R)$ *is not a planar graph.*

Proposition 7. (***The graph*** G) *Let* $R = R_\circ \oplus K^h$ *according to construction II. Then the Cayley graph denoted by G has p complete subgraphs. Moreover,*

(i) $| V(G) |= p(p^{1+hr})$.

(ii) G *is disconnected hence incomplete.*

(iii) G *is isomorphic to p copies of $K_{p(p^{hr})}$*

(iv) $diam(G) \le 2$.

(v) $gr(G) \le 3$

(vi) $b(G) \le 1$

(vii) $\Delta(G) = p((p^{hr}) - 1$.

(viii) $\chi(G) \le p(p^{hr})$.

(ix) $\omega(G) \le p(p^{hr})$

(x) $C_H(G) = 0$

(xi) G *obtained is non planar.*

Proof.   (i) From construction II, we have that $R_\circ = GR(p^{2r}, p^2)$ and $R = R_\circ \oplus U$. So U is an $R_\circ$-module generated by $\{u_1, \cdots, u_h\}$ elements where $h = 1$. It is clear that $Z(R) = R_\circ u_1$ and every element in G is of the form $(r_\circ, r_1)$.
Since
$$| R | = | R_\circ || U | = p(p^{1+hr}),$$
then $| V(G) | = p(p^{1+hr})$ which establishes (i).

(ii) Note that in Cayley graphs every element of R is a vertex and for any two elements $x, y \in R$ to be adjacent, $x - y \in (Z(R))$. Thus G has p complete disconnected subgraphs and hence, is incomplete.

(iii) This is clear Since by (ii), G is not complete. However we have p complete subgraphs of G, and each subgraph $G_n \cong K_{p(p^{hr})}$ such that $n = p$. This implies that $G \cong p$ copies of $k_{p(p^{hr})}$

(iv) $Diam(G) = sup\{d(x, y) | x, y \in R\}$ and that for all distinct $x, y \in R$, $x - y \in (Z(R))$. Now, G is having two complete interconnected subgraphs and so we have that $d(x, y) = 1$ for every subgraph. However, since there is an interconnection between the p subgraphs, the $sup\{d(x, y)\} = 2 \forall x, y \in G$. This implies that
$$diam(G) \le 2$$

(v) A complete graph G with n vertices is denoted that K_n. Now since G is having p copies of $K_{p(p^{rh})}$, subgraphs which are interconnected, the shortest cycle has three vertices and so the girth is 3

(vi) Let S be the set of vertices of minimal degree. By definition, $b(G) = \frac{|N(S)|}{|S|}$ taken over all $\Phi \ne S \subseteq V(G)$ such that $V(G) = N(S)$. In fact $S = N(S) \setminus Ann(Z(R))$. However the $Ann(Z(R)) = 0$, thus $| N(S) | = | S |$, hence the $b(G) = 1$.

(vii) $| V(G) | = p^{1+hr}$ and that (G) is not a complete graph. So let $u_i \in (G)$ be a vertex such that $u_i - u_j \in Z(R)$, we see that each vertex in each subgraph is adjacent to each other except itself for avoidance of loops. Therefore the number of vertices adjacent to u_i in each subgraph is $p(p^{hr}) - 1$. Thus $\Delta(G) = p(p^{hr}) - 1$.

(viii) Since $\Delta(G) = p(p^{hr}) - 1$ and each of the subgraphs of G is complete, then the minimum number of colors that can be assigned to each vertex is equivalent to the maximum degree of each vertex in (G) plus one since the vertex itself is also assigned a colour. Therefore it can be established that $\chi(G) = p(p^{rh})$.

(ix) The clique number of G is the order of the largest complete subgraph of G. Now, the subgraphs $G_1, G_2 \cdots G_n$ are the largest and of order $p(p^{hr})$, it then follows that $\omega(G) \le p(p^{hr})$.

(x) The definition of Harmonic centrality $H_G(u)$ of a vertex u and the Harmonic centralization $C_H(G)$, of a graph G is given in the proof of proposition 2.

Now from the graph G above, $m = | V |$.

We have that the Harmonic centrality of all the vertices is equal and thus the harmonic centralization is finally $= 0$.
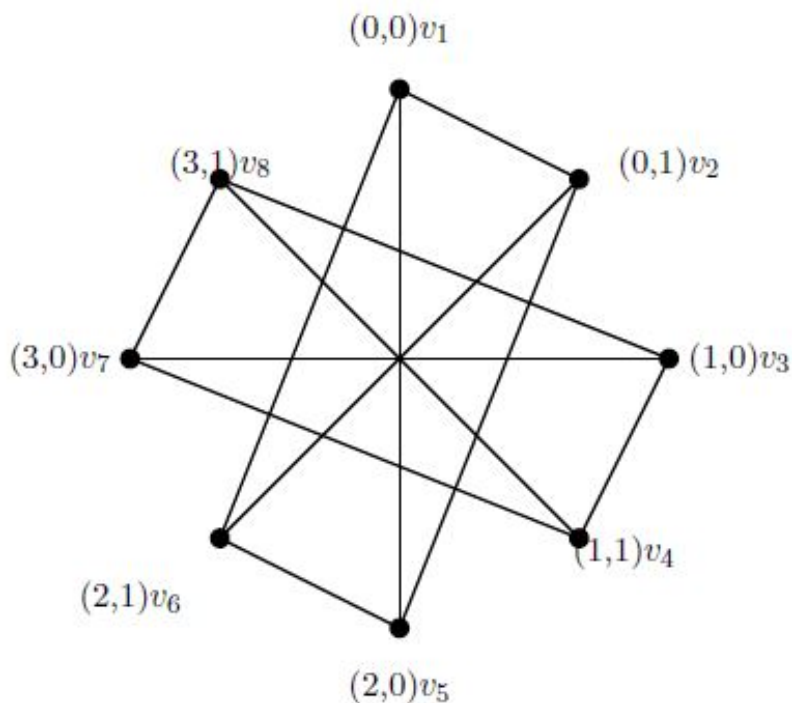
(xi) The graph G obtained has its edges intersecting within the plane. Thus, G is not planar.

$\square$

We provide some examples of Cayley graphs of rings of construction II.

Example 7. *Consider the ring R of construction II such that $p = 2$, $h = 1$, $k = 2$ and $r = 1$ so that $R = \mathbb{Z}_4 \oplus \mathbb{Z}_2$ so that*
$$R = \{(0, 1, 2, 3)(0, 1)\}$$
$$R = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)\}$$
$$Z(R) = \{(0, 0), (0, 1), (2, 0), (2, 1)\}$$

Therefore, G of this ring is represented by;

From the proposition above,G has the following properties:

Let $G_1 = \{v_1, v_2, v_5, v_6\}$ and $G_2 = \{v_3, v_4, v_7, v_8\}$, then ,
the $\mid V(G) \mid = 8$, $diam(G) = 2$, $girth(G) = 3$, $b(G) = 1$, $\Delta(G) = 3$, $\chi(G) = 4$, $\omega(G) = 4$,
$C_H(G) = 0$ and G obtained is not planar.

Example 8. *Consider the ring R of construction II such that $p = 3$, $h = 1$, $k = 2$ and $r = 1$ so that*

$$R = \mathbb{Z}_9 \oplus \mathbb{Z}_3$$

so that

$$R = \{(0, 1, 2, 3, 4, 5, 6, 7, 8)\} \oplus \{0, 1, 2\}$$

$$R = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2), (3,0), (3,1), (3,2), (4,0), (4,1), (4,2),$$
$$(5,0), (5,1), (5,2), (6,0), (6,1), (6,2), (7,0), (7,1), (7,2), (8,0), (8,1), (8,2)\}$$

$$Z(R) = \{(0,0), (0,1), (0,2), (3,0), (3,1), (3,2), (6,0), (6,1), (6,2)\}.$$

Therefore, G of this ring is represented by;
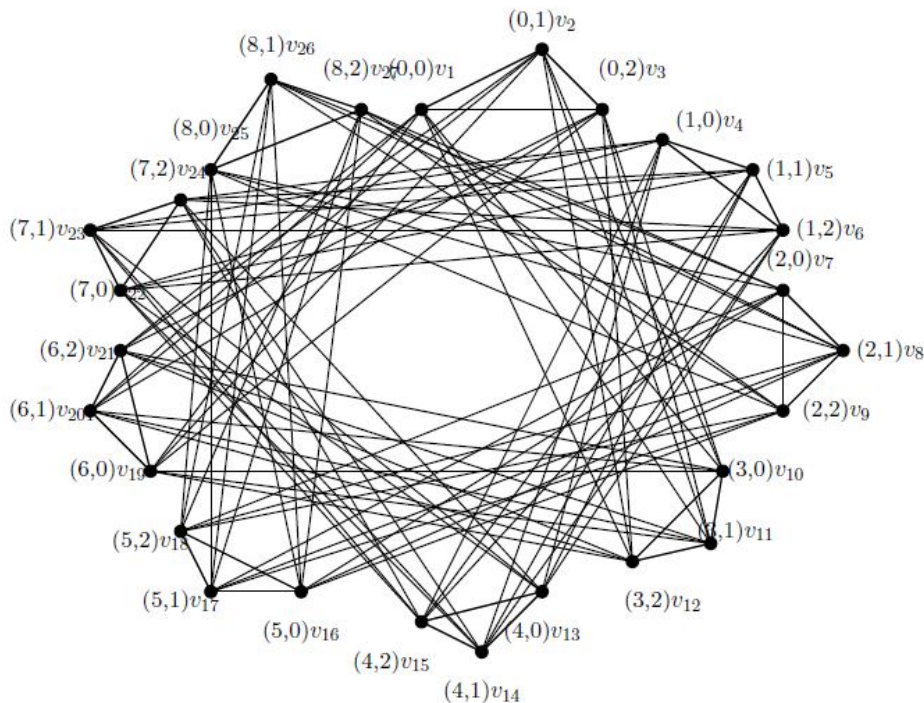
From the proposition above, $(CAY(R))$ has the following properties:

Let $G_1 = \{v_1, v_2, v_3, v_{10}, v_{11}, v_{12}, v_{19}, v_{20}, v_{21}\}$, $G_2 = \{v_4, v_5, v_6, v_{13}, v_{14}, v_{15}, v_{22}, v_{23}, v_{24}\}$, $G_3 = \{v_7, v_8, v_9, v_{16}, v_{17}, v_{18}, v_{25}, v_{26}, v_{27}\}$,
The $\mid V(G) \mid= 27$, $diam(G) = 2$, $girth(G) = 3$, $b(G) = 1$, $\Delta(G) = 8$, $\chi(G) = 9$, $\omega(G) = 9$, $C_H(G) = 0$ and G obtained is not planar.

4 The symmetric groups associated with the Zero divisor graphs for the square radical zero finite local rings above

Theorem 9. *Let R be a local ring of characteristic p with respect to the multiplication in construction I. Then*
$Aut(\Gamma(R)) \cong S_{p^{hr}-1}$ *and* $|Aut(\Gamma(R))| = (p^{hr}-2)! \sum_{i=1}^{hr} \phi(p^i)$ *where $\phi(p^i)$ is the Euler's-phi function of p^i $\forall i$.*

Proof. From construction I, we have that $R_\circ = GF(p^r, p)$, $R = R_\circ \oplus U$ and $Z(R) = R_\circ u_1 \oplus \cdots \oplus R_\circ u_h$. Every element in $Z(R) - 0$ is in the form $(0, x_1, \cdots, x_h)$ so that the product of every pair $(0, x_1, \cdots, x_h)(0, y_1, \cdots, y_h) = (0, \cdots, 0)$ implying that every elements(pairs) of $\Gamma(R)$ are adjacent.

$$\therefore |V(\Gamma(R))| = |Z(R) - 0| = p^{hr} - 1$$

$$\Rightarrow Aut(\Gamma(R)) \cong S_{p^{hr}-1},$$

the symmetric group on $p^{hr} - 1$ points. The orders of such groups are given by

$$|S_{p^{hr}-1}| = (p^{hr} - 1)! = (p^{hr} - 1)(p^{hr} - 2)! \cdots (i)$$

By definition:

$$\sum_{i=1}^{hr} \phi(p^i) = \phi(p^1) + \cdots + \phi(p^{hr})$$

$$= (p - 1) + p(p - 1) + p^2(p - 1) + \cdots + p^{hr-1}(p - 1)$$

$$= (p - 1)\{1 + p + p^2 + p^3 + \cdots + p^{hr-1}\}$$

$$= (p - 1)\frac{(p^{hr} - 1)}{(p - 1)} = p^{hr} - 1$$

Now

$$\frac{|Aut\Gamma(R)|}{\sum_{i=1}^{hr} \phi(p^i)} = \frac{(p^{hr} - 1)!}{(p^{hr} - 1)} = (p^{hr} - 2)!$$

$$\Rightarrow |Aut(\Gamma(R))| = (p^{hr} - 2)! \sum_{i=1}^{hr} \phi(p^i)$$

as required □

Theorem 10. *Let R be a ring of characteristic p^2 from classes of rings in constructionII. Then:*

(i)$| Aut\Gamma(R) |= (2p^{(h+1)r} - 3)! | E |$

(ii)$Aut(\Gamma(R)) \cong S_{p^{(h+1)r}-3} \times S_{p^{(h+1)r}-2} \times S_{p^{(h+1)r}-1}$

Proof. $Z(R)^* = \{x \in Z(R)\}\setminus\{0\} =| V\Gamma(R) |= p^{(h+1)r} - 1$ and therefore $| Aut\Gamma(R) |= (p^{(h+1)r} - 1)!$. But the sum of the degrees of $\Gamma(R)$ is $(p^{(h+1)r} - 1)(p^{(h+1)r} - 2)$.
Now let

$$\beta = \frac{1}{2}(p^{(h+1)r} - 1)(p^{(h+1)r} - 2)$$

and define formally

$$(p^{(h+1)r} - 1)! = (p^{(h+1)r} - 1)(p^{(h+1)r} - 2)(p^{(h+1)r} - 3)!$$

$$\Rightarrow (p^{(h+1)r} - 3)! = \frac{(p^{(h+1)r} - 1)!}{(p^{(h+1)r} - 1)(p^{(h+1)r} - 2)}$$

$$\Rightarrow 2(p^{(h+1)r} - 3)! = \frac{(p^{(h+1)r} - 1)!}{\beta}$$

$$\Rightarrow (p^{(h+1)r} - 1)! = 2(p^{(h+1)r} - 3)!\beta,$$

where $\beta =| E |$

$$\Rightarrow | Aut\Gamma(R) |= 2(p^{(h+1)r} - 3)! | E |,$$

which clears the first part of the proof.
The second part of the proof is easy. □

5   Conclusion

This paper has characterized the $2-$radical index of nilpotence finite local rings R given in constructions I and II using the structural, geometric and algebraic properties of the Zero divisor and Cayley graphs of R. It is evident from the main results that the graphs $\Gamma(R)$ of power two radical zero local rings of characteristic p and p^2 are complete graphs of order $p^{rh}-1$. They are also Hamiltonian graphs and as the value of p increases, the more the hamiltonian cycles. The Cayley graphs on the other hand are incomplete with complete subgraphs which are copies of each other. Moreover, unlike Zero divisor graphs, Cayley graphs represent noisy geometries such that one cannot easily describe the algebraic and the structural properties of the whole graph.

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Competing Interests

Authors have declared that no competing interests exist.

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