Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 6385673, 12 pages
https://doi.org/10.1155/2022/6385673

Research Article

Computation of the Fault-Tolerant Metric Dimension of Certain Networks

Humera Bashir, Zohaib Zahid, and Michael Onyango Ojiema

University of Management and Technology (UMT), Lahore, Pakistan

Correspondence should be addressed to Michael Onyango Ojiema; mojiema@mmust.ac.ke

Received 12 December 2021; Accepted 23 February 2022; Published 21 March 2022

Academic Editor: Muhammad Faisal Nadeem

1. Introduction and Preliminaries

The notion of metric dimension originated in the twentieth century by the work done by Slater [1, 2] and latter independently by Harary and Melter [2]. The initial concept was introduced to locate the intruder in any network but the development and impact of the notion was far reaching in coming years. Several applications of metric related network parameters can be seen in social networking, navigation, communications, and engineering, pharmaceutical chemistry. The fault tolerance of a networking system is its ability to perform in case of malfunction of one of the source nodes. The fault tolerant metric dimension of a network can be regarded as a process of uniquely identification of each node of a network in case a malfunction occurs in one of the source node. This allows fault tolerant metric dimension more adaptability and flexibility for practical purposes than the parent notion.

For a simple network $N = (V(N), E(N))$, the size of least distant route between any two of its nodes s, w is referred as distance between them, notated as $d(s, w)$. If $\mathfrak{R} = \{n_1, n_2, \ldots, n_t\}$ is an arrayed collection of nodes in N then the vector $(d(n, n_i))_{i=1}^{t}$ is called a representative vector of the node n wrt. $\mathfrak{R}$ notated as $r(n|\mathfrak{R})$. The collection $\mathfrak{R}$ containing distinct representative vectors corresponding to distinct nodes is referred as resolving collection of nodes. A resolving collection of nodes with least members is called basis for the network N and its size is referred as metric dimension of N, represented by $\dim(N)$.

The limitations in the parent notion of metric dimension of having deviation at only one position between the representative tuples of vertices led Estrado-Moreno et al., to introduce a generalized notion of $k-$ metric dimension in 2014 (see [3]). Thus, the notion of the $k-$ metric dimension was introduced, which becomes the metric dimension, for $k = 1$, denoted by $\dim(N)$. The equivalent condition on the existence of $k-$ metric basis and related results were presented by Estrado-Moreno et al. in [4]. Further the notion was explored in context of generalized metric spaces in [6] and respectively for lexicographic product of graphs and corona product of graphs, in [5, 6]. The extension of this concept to the more general case of non-necessarily connected graphs is studied in [7]. The complexity of some $k-$ metric dimension problems revealed that its computation is NP-hard (see [8]). This motivated the study of notion $k-$ metric dimension problems for particular values of k. The notion for $k = 2$, is referred as fault tolerant metric dimension, which was introduced by Hernando in 2008 (see [9]). Here, we include a formal definition of the notion: Let $\mathfrak{R} = \{n_1, n_2, \ldots, n_l\}$ be an arrayed collection of nodes in N. If for each pair of nodes $a, b \in V(N)$ their absolute difference representation $AD((a,b)|\mathfrak{R}) = (|d_N(a, n_1) - d_N(b, n_1)|, \ldots, |d_N(a, n_l) - d_N(b, n_l)|)$ contains more than one zeros, then the collection $\mathfrak{R}$ is referred as a fault-tolerant resolving set (FTRS) for N. An FTRS of the least size in N is called as fault tolerant metric basis (FTMB) and its size is the fault tolerant metric dimension (FTMD) of N, notated by $\wp(N)$.

The notion FTMD have been extensively discussed by several researchers. In this regard, FTMD for prism related graphs and circulant graphs have been studies in [10, 11], FTMR of lexicographic product and some other classes have been discussed in [12, 13], FTMD of certain wheel related graphs can be seen in [14]. Further, FTMD for convex polytopes and triangular lattices are computed in [15, 16]. Recently, in 2020, Huo et al. computed FTMD for generalized prism graph and Mobious ladders and latter in 2021 Bashir et al. discussed FTMD of some classes of rotationally symmetric graphs (see [17, 18]). Some other related developments can also be seen in [19, 20]. Following are the two theorems which will be helpful in computing our main results.

Theorem 1 (see [3]). *For any graph $G \dim(G) < \wp(G)$.*

Theorem 2 (see [3]). *If $G \neq P_n$, then $\wp(G) \geq 3$.*

The symmetric planer graphs, like generalized Petersen and sunlet networks have key importance in the fields of telecommunication, navigation and networking due to the structure of these networks which results in uniform rate of data transfer and thereby optimizing the resources used.

1.1. Main Results. The study conducted in this article, lead to following main results

Theorem 3

(1) For $n \geq 3$,

$$\wp(S_n) = \begin{cases} 3, & \text{if } n \text{ is } 3, 5, 7 \text{ or } 9, \\ 4, & \text{else.} \end{cases} \tag{1}$$

(2) For $n \geq 3$,

$$\wp(P(n,1)) = \begin{cases} 3, & \text{if } n = 3, \\ 4, & \text{else.} \end{cases} \tag{2}$$

(3) For $n \geq 5$,

$$\wp(P(n,2)) = \begin{cases} 4, & \text{for even } n, \\ 4 \text{ or } 5, & \text{for odd } n. \end{cases} \tag{3}$$

The rest of the article is organized in the following manner: In Section 2, the FTMD of family of $n-$ sunlet graph S_n is computed. The Section 3 comprises of the computation of FTMD of family the generalized Petersen graph $P(n,t)$, for $t = 1$. We also computed the FTMD of $P(n,2)$ for even n and some tight bounds are obtained for odd n. An application of the current work in context of navigational routing problem is furnished in Section 4. Lastly, the paper is concluded with open problems in Section 5.

2. The FTMD of family of the $n-$ sunlet graph

The family of $n-$ sunlet graph denoted by S_n is the graph obtained by attaching n pendant edges to a cycle graph C_n as shown in Figure 1. The vertex set $V(S_n) = \{u_i, v_i | 1 \leq i \leq n\}$ and edge set $E(S_n) = \{u_i u_{i+1}, u_i v_i | 1 \leq i \leq n\}$, where subscripts are to be read modulo n. In the following lemma, the metric dimension of the family of $n-$ sunlet graph S_n is presented.

Lemma 1. *The metric dimension of S_n, for $n \geq 3$ is*

$$\dim(S_n) = \begin{cases} 2, & \text{if } n \text{ is odd or } n = 4, \\ 3, & \text{else.} \end{cases} \tag{4}$$

Proof. In order to prove the theorem, following cases can be considered: □

Case 1. (When n is odd)

Let $n = 2m + 1$, for $m \geq 1$ and take $\mathfrak{R} = \{u_1, u_{(n+1/2)}\}$. Representation of the vertices u_i and v_i with respect to $\mathfrak{R}$ is shown in Table 1.

We can see that for all $a, b \in V(S_n)$, $r(a|\mathfrak{R}) \neq r(b|\mathfrak{R})$. Hence, $\mathfrak{R}$ is a resolving set. This implies that $\dim(S_n) \leq 2$. Since, $\dim(G) = 1$ if and only if G is a path graph, therefore, $\dim(S_n) = 2$.

Case 2. (When $n = 4$)

It can be easily verified that $\mathfrak{R} = \{v_1, v_2\}$ is a minimal resolving set for S_n. Therefore, we have $\dim(S_n) = 2$.

Case 3. (When n is even and $n \geq 6$)

Let $n = 2m$, for $m \geq 3$ and take $\mathfrak{R} = \{u_1, u_2, u_{(n+2/2)}\}$. Representation of the vertices u_i and v_i with respect to $\mathfrak{R}$ is shown in Table 2.

We can see that for all $a, b \in V(S_n)$, $r(a|\mathfrak{R}) \neq r(b|\mathfrak{R})$. Hence, $\mathfrak{R}$ is a resolving set. Therefore, $\dim(S_n) \leq 3$. It is exclusively required to prove that $\dim(S_n) \neq 2$, for $n \geq 6$. This can be achieved by showing that S_n is unable to have a resolving set of order 2, leading to the following possibilities:

a) If $\mathfrak{R} = \{u_1, u_i\}$ with $1 < i \leq m$, then $r(u_n|\mathfrak{R}) = r(v_1|\mathfrak{R})$.

b) If $\mathfrak{R} = \{v_1, v_i\}$, then $r(u_{n-1}|\mathfrak{R}) = \begin{cases} r(v_n|\mathfrak{R}), & \text{if } 1 < i < m, \\ r(v_2|\mathfrak{R}), & \text{if } i = m. \end{cases}$

c) If $\mathfrak{R} = \{u_1, v_i\}$, then $r(u_{n-1}|\mathfrak{R}) = \begin{cases} r(v_n|\mathfrak{R}), & \text{if } 1 \leq i < m, \\ r(v_2|\mathfrak{R}), & \text{if } i = m. \end{cases}$

d) If $\mathfrak{R} = \{v_1, u_i\}$, then $r(u_{n-1}|\mathfrak{R}) = \begin{cases} r(v_n|\mathfrak{R}), & \text{if } 1 \leq i < m, \\ r(v_2|\mathfrak{R}), & \text{if } i = m. \end{cases}$

Hence, in all above possibilities, it is concluded that S_n does not have a 2 cardinality resolving set. This concludes the proof. □

The above lemma will be helpful in the following result.

Theorem 4. *The FTMD of S_n, for $n \geq 3$ is*

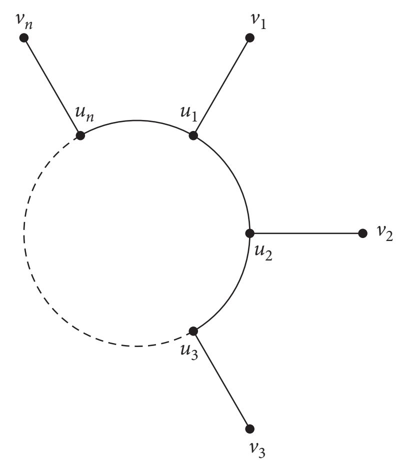

FIGURE 1: $n-$ sunlet graph S_n.

TABLE 1: Representation of nodes in S_n for even n.

i	$r(u_i\|\mathfrak{R})$	i	$r(v_i\|\mathfrak{R})$
$2 \le i \le m$	$(i-1, m-i+1)$	$1 \le i \le m+1$	$(i, m-i+2)$
$m+2 \le i \le n$	$(n-i+1, i-m-1)$	$m+2 \le i \le n$	$(n-i+2, i-m)$

TABLE 2: Representation of nodes in S_n for even $n \ge 6$.

i	$r(u_i\|\mathfrak{R})$	i	$r(v_i\|\mathfrak{R})$
$3 \le i \le m$	$(i-1, i-2, m-i+1)$	$1 \le i \le m+1$	$(i, i-1, m-i+2)$
$m+2 \le i \le n$	$(n-i+1, n-i+2, i-m-1)$	$m+2 \le i \le n$	$(n-i+2, n-i+3, i-m)$

$$\wp(S_n) = \begin{cases} 3, & \text{if } n \text{ is } 3, 5, 7 \text{ or } 9, \\ 4, & \text{else.} \end{cases} \quad (5)$$

Proof. In order to show the assertion, following cases can be considered: □

Case 1. (When $n = 3, 5, 7$ or 9)

It can be immediately confirmed that $\mathfrak{R} = \{v_1, v_2, v_3\}$, when $n = 3$ and $\mathfrak{R} = \{v_1, v_{(n-1/2)}, v_{n-2}\}$, when $n = 5, 7, 9$, are FTRSs for S_n. This implies that $\wp(S_n) \le 3$. Since, by Lemma 1, S_n has metric dimension 2 (when n is odd), therefore, by using Theorem 1, it is clear that $\wp(S_n) \ge 3$. This implies that $\wp(S_n) = 3$.

Case 2. (When $n = 4$)

Take $\mathfrak{R} = \{u_1, u_2, u_3, u_4\}$. Then the representation of the vertices u_i and v_i with reference to the above said set $\mathfrak{R}$ are

$$r(u_i|\mathfrak{R}) = \begin{cases} (0,1,2,1), & \text{if} i = 1, \\ (1,0,1,2), & \text{if} i = 2, \\ (2,1,0,1), & \text{if} i = 3, \\ (1,2,1,0), & \text{if} i = 4, \end{cases} \quad \text{and}$$

$$r(v_i|\mathfrak{R}) = \begin{cases} (1,2,3,2), & \text{if} i = 1, \\ (2,1,2,3), & \text{if} i = 2, \\ (3,2,1,2), & \text{if} i = 3, \\ (2,3,2,1), & \text{if} i = 4. \end{cases}$$

We can observe that more than one zeros exist in the $AD((a,b)|\mathfrak{R})$, for each $a, b \in V(S_4)$. Hence, $\mathfrak{R}$ is a FTRS. Therefore, in view of Lemma 1 togather with Theorem 1, we conclude that $\dim(S_4) = 2 < \wp(S_4) \le 4$. The only thing that remains to show is $\wp(S_4) \ne 3$. The Table 3 shows that S_4 has no FTRS of cardinality 3. Hence, $\wp(S_4) = 4$.

Case 3. (When n is even and $n \ge 6$)

Let $n = 2m$, for $m \ge 3$ and take $\mathfrak{R} = \{u_1, u_2, u_{m+1}, u_{m+2}\}$. Representation of the vertices u_i and v_i with respect to $\mathfrak{R}$ is shown in Table 4.

We can observe that more than one zeros exist in the $AD((a,b)|\mathfrak{R})$, for each $a, b \in V(S_n)$. Hence, $\mathfrak{R}$ is a FTRS. Now, by Lemma 1 and Theorem 1, we have $\wp(S_n) = 4$.

Case 4. (When n is odd and $n \ge 11$)

Let $n = 2m+1$ with $m \ge 5$ and take $\mathfrak{R} = \{u_1, u_2, u_{m+1}, u_{m+2}\}$. Representation of the vertices u_i and v_i with respect to $\mathfrak{R}$ is shown in Table 5.

Therefore, in lights of Lemma 1 combined with the Theorem 1, we have $\dim(S_n) = 2 < \wp(S_n) \le 4$. The only thing that remains to show is $\wp(S_n) \ne 3$. In order to achieved that S_n is unable to have a FTRS of order 3, we have the following possibilities:

a) If $\mathfrak{R} = \{u_1, u_i, u_j\}$ such that $i, j \in \{2, 3, \ldots, m+1\}$ and $i < j$, then $AD((u_1, v_2)|\mathfrak{R}) = (2, 0, 0)$.

Table 3: 3– subsets of $V(S_4)$ and pair of vertices where absolute difference representation has atleast two zeros.

$\mathfrak{R}$	(x, y)	$AD((x,y)\vert\mathfrak{R})$
$\{u_1, u_2, u_i\}$ with $2 < i \leq 4$	(u_4, v_1)	$(0,0,2)$
$\{u_1, u_3, u_4\}$	(u_2, u_4)	$(0,2,0)$
$\{v_1, v_2, v_3\}$	(u_2, u_4)	$(2,0,0)$
$\{v_1, v_2, v_4\}$	(u_1, u_3)	$(0,0,2)$
$\{v_1, v_3, v_4\}$	(u_2, u_4)	$(0,0,2)$
$\{u_1, v_1, v_i\}$ with $1 < i \leq 4$	(u_2, u_4)	$(0,0,2)$
$\{u_1, v_2, v_i\}$ with $2 < i \leq 4$	(u_4, v_1)	$(0,0,2)$
$\{u_1, v_3, v_4\}$	(u_2, u_4)	$(0,0,2)$
$\{v_1, u_1, u_i\}$ with $1 < i \leq 4$	(u_2, u_4)	$(0,0,2)$
$\{v_1, u_2, u_3\}$	(u_1, v_2)	$(2,0,0)$
$\{v_1, u_2, u_4\}$	(u_1, u_3)	$(2,0,0)$
$\{v_1, u_3, u_4\}$	(u_2, v_3)	$(2,0,0)$

Table 4: Representation of nodes in S_n for even $n \geq 6$.

i	$r(u_i\vert\mathfrak{R})$	i	$r(v_i\vert\mathfrak{R})$
$i = 1$	$(0, 1, m, m-1)$	1	$(1, 2, m+1, m)$
$2 \leq i \leq m$	$(i-1, i-2, m+1-i, m+2-i)$	$2 \leq i \leq m+1$	$(i, i-1, m+2-i, m+3-i)$
$i = m+1$	$(m, m-1, 0, 1)$	$i = m+2$	$(m, m+1, 2, 1)$
$m+2 \leq i \leq n$	$(n-i+1, n-i+2, i-m-1, i-m-2)$	$m+3 \leq i \leq n$	$(n-i+2, n-i+3, i-m, i-m-1)$

Table 5: Representation of nodes in S_n for odd $n \geq 11$.

i	$r(u_i\vert\mathfrak{R})$	i	$r(v_i\vert\mathfrak{R})$
$i = 1$	$(0, 1, m, m)$	1	$(1, 2, m+1, m+1)$
$2 \leq i \leq m+1$	$(i-1, i-2, m+1-i, m+2-i)$	$2 \leq i \leq m+1$	$(i, i-1, m+2-i, m+3-i)$
$i = m+2$	$(m, m, 0, 1)$	$i = m+2$	$(m+1, m+1, 2, 1)$
$m+3 \leq i \leq n$	$(n-i+1, n-i+2, i-m-1, i-m-2)$	$m+3 \leq i \leq n$	$(n-i+2, n-i+3, i-m, i-m-1)$

b) If $\mathfrak{R} = \{u_1, u_i, u_j\}$ such that $i, j \in \{2, 3, \ldots, n\}$ and $i \leq m < j$, then

$$AD((u_n, v_1)|\mathfrak{R}) = \begin{cases} (0,0,1), & \text{if } j = m+1, \\ (0,0,2), & \text{else.} \end{cases} \tag{6}$$

For pendant vertices v_i, consider $\mathfrak{R} = \{v_1, v_i, v_j\}$ with $1 < i < j \leq m+1$ and $1 \leq i \leq m < j \leq n$. The proofs are similar as Cases (a) and (b) respectively.

c) If $\mathfrak{R} = \{u_1, v_i, v_j\}$, such that $i, j \in \{1, 2, \ldots, m+1\}$ and $i < j$, then $AD((u_{n-1}, v_n)|\mathfrak{R}) = (0,0,2)$ for $i = 1$, $j = m+1$ and

$$AD((u_n, v_1)|\mathfrak{R}) = \begin{cases} (0,2,0), & \text{if } i = 1, j \neq m+1, \\ (0,0,1), & \text{if } i \neq 1, j = m+1, \\ (0,0,0), & \text{else.} \end{cases} \tag{7}$$

d) If $\mathfrak{R} = \{u_1, v_i, v_j\}$, such that $i, j \in \{2, 3, \ldots, n\}$ and $i \leq m < j$, then

$$AD((u_n, v_1)|\mathfrak{R}) = \begin{cases} (0,0,1), & \text{if } j = m+1, \\ (0,0,2), & \text{else,} \end{cases} \tag{8}$$

and if $\mathfrak{R} = \{u_1, v_1, v_j\}$, with $m < j \leq n$, then $AD((u_{n-1}, v_n)|\mathfrak{R}) = (0,0,2)$.

e) If $\mathfrak{R} = \{v_1, u_i, u_j\}$, such that $i, j \in \{1, 2, \ldots, m+1\}$ and $i < j$, then

$$AD((v_n, u_{n-1})|\mathfrak{R}) = \begin{cases} (0,0,0), & \text{if } 1 \leq i < j \leq m-1, \\ (0,0,2), & \text{if } 1 \leq i \leq m-1, m \leq j \leq m+1, \end{cases} \tag{9}$$

and $AD((v_{m+1}, u_{m+2})|\mathfrak{R}) = (1,0,0)$ for $i = m$, $j = m+1$.

f) If $\mathfrak{R} = \{v_1, u_i, u_j\}$, such that $i, j \in \{1, 2, \ldots, n\}$ and $i \leq m < j$, then

$$AD((u_{n-1}, v_n)|\mathfrak{R}) = \begin{cases} (0,0,1), & \text{for } j = m, \\ (0,0,0), & \text{for } j = n, \\ (0,0,2), & \text{otherwise.} \end{cases} \tag{10}$$

Hence, in all above possibilities, we conclude that there is no FTRS for S_n containing exactly 3 nodes. This concludes the proof. □

3. The FTMD of family of The Generalized Petersen graphs $P(n, t)$

Coxeter was the first one to introduce the generalized notion of Petersen graphs $P(n, t)$ in 1950 (see [23]). It is an important class of graphs with vertex set $V(P(n,t)) = \{u_i, v_i | 1 \le i \le n\}$ and edge set $E(P(n,t)) = \{v_i v_{i+1}, u_i v_i, u_i u_{i+t} | 1 \le i \le n\}$, where subscripts are to be read modulo n and $1 \le t \le (n-1/2)$. The Petersen graphs $P(8,1)$ and $P(8,2)$ are shown in Figures 2 and 3 respectively.

The forthcoming lemma will be resourceful in the computation the FTMD in regards to family of the generalized Petersen graph $P(n,1)$.

Lemma 2 (see [24]). *For the generalized Petersen graph $P(n,1)$;*

$$\dim(P(n,1)) = \begin{cases} 2, & \text{for odd } n, \\ 3, & \text{for even } n. \end{cases} \tag{11}$$

Theorem 5. *The FTMD of $P(n,1)$, for $n \ge 3$ is*

$$\wp(P(n,1)) = \begin{cases} 3, & \text{if } n = 3, \\ 4, & \text{else.} \end{cases} \tag{12}$$

Proof. In order to show the assertion, following cases can be considered: □

Case 1. (When n is even)

Take $\mathfrak{R} = \{u_1, u_2, v_1, v_2\}$, then for the vertices u_i and v_i the representation is shown in Table 6.

Since it can be observed that more than one zeros exist in the $AD((a,b)|\mathfrak{R})$, for each $a, b \in V(P(n,1))$. Therefore, $\wp(P(n,1)) \le 4$. Hence, in view of Lemma 2 together with the Theorem 1, we conclude that $\wp(P(n,1)) = 4$.

Case 2. (When n is odd)

This case is further subdivided as follows:

Case 2a. (When $n = 3$)

For $P(3,1)$ is can be confirmed immediately that $\mathfrak{R} = \{u_1, u_2, u_3\}$ is its FTRS. This implies that $\wp(P(3,1)) \le 3$. Therefore, in view of the Lemma 2 and the Theorem 1, we conclude that $\wp(P(3,1)) = 3$.

Case 2b. (When n is odd and $n \ge 5$)

Let $n = 2m+1$, for $m \ge 1$ and take $\mathfrak{R} = \{u_1, u_2, v_1, v_2\}$, then for the vertices u_i and v_i the representation is shown in Table 7.

Since it can be observed that more than one zeros exist in the $AD((a,b)|\mathfrak{R})$, for each $a, b \in V(P(n,1))$. Therefore, $\wp(P(n,1)) \le 4$. Hence, in view of Lemma 3 and Theorem 1,

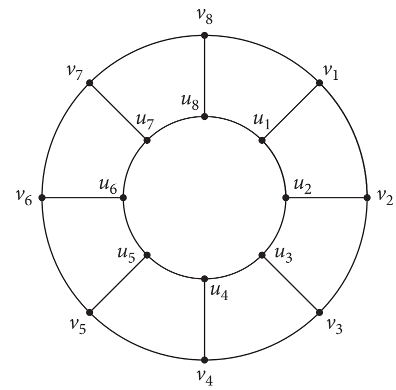

Figure 2: The Petersen graph $P(8,1)$.

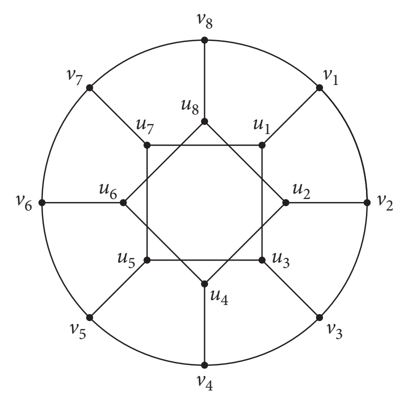

Figure 3: The Petersen graph $P(8,2)$.

we have $\dim(P(n,1)) = 2 < \wp(P(n,1)) \le 4$. It is exclusively required to prove that $\wp(P(n,1)) \ne 3$. This can be achieved by showing that $P(n,1)$ is unable to have a resolving set of order 3, leading to the following possibilities:

a) If $\mathfrak{R} = \{u_1, u_i, u_j\}$ such that $i, j \in \{2, 3, \ldots, m+1\}$ and $i < j$, then $AD((u_1, v_2)|\mathfrak{R}) = (2,0,0)$.

b) If $\mathfrak{R} = \{u_1, u_i, u_j\}$ such that $i, j \in \{2, 3, \ldots, n\}$ and $i \le m < j$, then

$$AD((u_n, v_1)|\mathfrak{R}) = \begin{cases} (0,0,1), & \text{if } j = m+1, \\ (0,0,2), & \text{else.} \end{cases} \tag{13}$$

c) If $\mathfrak{R} = \{u_1, v_i, v_j\}$, such that $i, j \in \{1, 2, \ldots, m\}$ and $i < j$, then $AD((u_1, v_n)|\mathfrak{R}) = (2,0,0)$ and if $j = m+1$, then

$$AD((v_1, u_2)|\mathfrak{R}) = \begin{cases} (0,2,0), & \text{if } i = 1, \\ (0,0,0), & 2 \le i \le m. \end{cases} \tag{14}$$

d) If $\mathfrak{R} = \{u_1, v_i, v_j\}$, such that $i, j \in \{1, 2, \ldots, n\}$ and $i \le m < j$, then

Table 6: Representation of nodes in $P(n,1)$ for even n.

i	$r(u_i\|\mathfrak{R})$	$r(v_i\|\mathfrak{R})$
1	$(0,1,1,2)$	$(1,2,0,1)$
$2\leq i\leq m+1$	$(i-1,i-2,i,i-1)$	$(i,i-1,i-1,i-2)$
$m+2\leq i\leq n$	$(n-i+1,n-i+2,n-i+2,n-i+3)$	$(n-i+2,n-i+3,n-i+1,n-i+2)$

Table 7: Representation of nodes in $P(n,1)$ for odd $n\geq 5$.

i	$r(u_i\|\mathfrak{R})$	$r(v_i\|\mathfrak{R})$
1	$(0,1,1,2)$	$(1,2,0,1)$
$2\leq i\leq m+1$	$(i-1,i-2,i,i-1)$	$(i,i-1,i-1,i-2)$
$i=m+2$	$(m,m,m+1,m+1)$	$(m+1,m+1,m,m)$
$m+3\leq i\leq n$	$(n-i+1,n-i+2,n-i+2,n-i+3)$	$(n-i+2,n-i+3,n-i+1,n-i+2)$

Table 8: Representation of nodes in $P(n,2)$ for $n\equiv 0 \bmod 4\, n\geq 8$ and even i.

i (even)	$r(u_i\|\mathfrak{R})$	$r(v_i\|\mathfrak{R})$
$i=2$	$(3,3,(m+4/2),(m+4/2))$	$(2,2,(m+2/2),(m+2/2))$
$4\leq i\leq m$	$((i+4/2),(i+2/2),(m-i+6/2),(m-i+8/2))$	$((i+2/2),(i/2),(m-i+4/2),(m-i+6/2))$
$i=m+2$	$((m+4/2),(m+4/2),3,3)$	$((m+2/2),(m+2/2),2,2)$
$m+4\leq i\leq n$	$((n-i+6/2),(n-i+8/2),(i-m+4/2),(i-m+2/2))$	$((n-i+4/2),(n-i+6/2),(i-m+2/2),(i-m/2))$

Table 9: Representation of nodes in $P(n,2)$ for $n\equiv 0 \bmod 4\, n\geq 8$ and odd i.

i (odd)	$r(u_i\|\mathfrak{R})$	$r(v_i\|\mathfrak{R})$
1	$(0,1,(m/2),(m-2/2))$	$(1,2,(m+2/2),(m/2))$
$3\leq i\leq m-1$	$((i-1/2),(i-3/2),(m-i+1/2),(m-i+3/2))$	$((i+1/2),(i-1/2),(m-i+3/2),(m-i+5/2))$
$i=m+1$	$((m/2),(m-2/2),0,1)$	$((m+2/2),(m/2),1,2)$
$m+3\leq i\leq n-1$	$((n-i+1/2),(n-i+3/2),(i-m-1/2),(i-m-3/2))$	$((n-i+3/2),(n-i+5/2),(i-m+1/2),(i-m-1/2))$

Table 10: Representation of nodes in $P(n,2)$ for $n\equiv 2 \bmod 4$ and $n\geq 10$ and even i.

i (even)	$r(u_i\|\mathfrak{R})$	$r(v_i\|\mathfrak{R})$
$i=2$	$(3,3,(m+5/2),(m+3/2))$	$(2,2,(m+3/2),(m+1/2))$
$4\leq i\leq m+1$	$((i+4/2),(i+2/2),(m-i+7/2),(m-i+9/2))$	$((i+2/2),(i/2),(m-i+5/2),(m-i+7/2))$
$i=m+3$	$((m+3/2),(m+5/2),3,3)$	$((m+1/2),t(m+3/2)n,q2,2))$
$m+5\leq i\leq n$	$((n-i+6/2),(n-i+8/2),(i-m+3/2),(i-m+1/2))$	$((n-i+4/2),(n-i+6/2),(i-m+1/2),(i-m-1/2))$

Table 11: Representation of nodes in $P(n,2)$ for $n\equiv 2 \bmod 4$ and $n\geq 10$ and odd i.

i (odd)	$r(u_i\|\mathfrak{R})$	$r(v_i\|\mathfrak{R})$
1	$(0,1,(m-1/2),(m-3/2))$	$(1,2,(m+1/2),(m-1/2))$
$i=3$	$(1,0,(m-1/2),(m-1/2))$	$(2,1,(m+1/2),(m+1/2))$
$5\leq i\leq m$	$((i-1/2),(i-3/2),(m-i+2/2),(m-i+4/2))$	$((i+1/2),(i-1/2),(m-i+4/2),(m-i+6/2))$
$i=m+2$	$((m-1/2),(m-1/2),0,1)$	$((m+1/2),(m+1/2),1,2)$
$i=m+4$	$((m-3/2),(m-1/2),1,0)$	$((n-i+3/2),(n-i+5/2),(i-m/2),(i-m-2/2))$
$m+6\leq i\leq n$	$((n-i+1/2),(n-i+3/2),(i-m-2/2),(i-m-4/2))$	$((n-i+3/2),(n-i+5/2),(i-m/2),(i-m-2/2))$

$$AD((v_1,u_2)|\mathfrak{R}) = \begin{cases} (0,0,0), & \text{if } i\neq 1,\ j=m+1, \\ (0,0,2), & \text{if } i\neq 1,\ j\neq m+1, \\ (0,2,0), & \text{if } i=1,\ j=m+1, \end{cases} \tag{15}$$

and if $i=1$ and $j\neq m+1$ then $AD((u_n,v_1)|\mathfrak{R}) = (0,2,0)$.

Therefore, all the aforesaid cases reveal that $P(n,1)$ have not FTRS of cardinality 3. This concludes the proof. □

The upcoming theorem will be aided by the following lemma.

Table 12: Representation of nodes in $P(n, 2)$ for $n \equiv 1 \bmod 4$ and $n \geq 9$ and even i.

i (even)	$r(u_i \mid \mathfrak{R})$	$r(v_i \mid \mathfrak{R})$
$i = 2$	$(4, 1, (m+2/2), (m+2/2), 2)$	$(3, 2, (m+4/2), (m+4/2), 2)$
$4 \leq i \leq m-4$	$((i+6/2, (i/2)), (m-i+4/2), (m-i+6/2), (i+2/2))$	$((i+4/2), (i+2/2), (n-i-5/2), (n-i-3/2), (i+4/2))$
$i = m-2$	$((m+2/2), (m-2/2), 3, 4, (m/2))$	$((i+4/2), (i+2/2), (n-i-5/2), (n-i-3/2), (i+4/2))$
$i = m$	$((n-m-1/2), (m/2), 2, 3, (m+2/2))$	$((m+2/2), (m+2/2), 2, 3, (m+4/2))$
$i = m+2$	$((n-m-3/2), (m+2/2), 1, 2, (m+2/2))$	$((m/2), (m+2/2), 0, 1, (m+4/2))$
$i = m+4$	$((n-i-1/2), (n-i+5/2), (i-m/2), (i-m/2), (n-i+3/2))$	$((n-m-3/2), (n-m-1/2), 2, 1, (n-m+1/2))$
$i = m+6$	$((n-i-1/2), (n-i+5/2), (i-m/2), (i-m/2), (n-i+3/2))$	$((n-m-5/2), (n-m-3/2), 4, 3, n-m-6)$
$m+8 \leq i \leq n-3$	$((n-i-1/2), (n-i+5/2), (i-m/2), (i-m/2), (n-i+3/2))$	$((n-i+1/2), (n-i+3/2), (i-m+2/2), (i-m+2/2), (n-i+3/2))$
$i = n-1$	$(0, 3, (m/2), (m+2/2), 2)$	$(1, 2, (m+2/2), (m+2/2), 1)$

Table 13: Representation of nodes in $P(n, 2)$ for $n \equiv 1 \bmod 4$ and $n \geq 9$ and odd i.

i (odd)	$r(u_i \vert \mathfrak{R})$	$r(v_i \vert \mathfrak{R})$
$i = 1$	$(1, 3, (m+2/2), (m+2/2), 2)$	$(2, 2, (m+4/2), (m+4/2), 1)$
$3 \leq i \leq m-3$	$((i+1/2), (i+5/2), (n-i-6/2), (n-i-6/2), (i+3/2))$	$((i+3/2), (i+3/2), (n-i-4/2), (n-i-4/2), (i+5/2))$
$i = m-1$	$((m/2), (m+2/2), (n-m-7/2), (n-m-7/2), (m+2/2))$	$((i+3/2), (i+3/2), (n-i-4/2), (n-i-4/2), (i+5/2))$
$i = m+1$	$((m+2/2), (m/2), 2, 2, (m+2/2))$	$((m+2/2), (m+2/2), 1, 2, (n-m+3/2))$
$i = m+3$	$((n-i+4/2), (n-i/2), (i-m+1/2), (i-m-1/2), (n-i+2/2))$	$(m+2/2), (m/2), 1, 0, (m+4/2)$
$i = m+5$	$((n-i+4/2), (n-i/2), (i-m+1/2), (i-m-1/2), (n-i+2/2))$	$((n-m-3/2), (n-m-3/2), 3, 2, (n-m-1/2))$
$m+7 \leq i \leq n-2$	$((n-i+4/2), (n-i/2), (i-m+1/2), (i-m-1/2), (n-i+2/2))$	$((n-i+2/2), (n-i+2/2), (i-m+3/2), (i-m+1/2), n-i)$
$i = n$	$(3, 0, (m+2/2), (m/2), 1)$	$(2, 1, (m+2/2), (m+2/2), 0)$

Table 14: Representation of nodes in $P(n, 2)$ for $n \equiv 3 \bmod 4$ and $n \geq 11$ and even i.

i (even)	$r(u_i \mid \Re)$	$r(v_i \mid \Re)$
$i = 2$	$((i + 6/2), (i/2), (n - i - 7/2), (n - i - 7/2), (i + 4/2))$	$(3, 2, (m + 5/2), (m + 5/2), 3)$
$4 \leq i \leq m - 3$	$((i + 6/2), (i/2), (n - i - 7/2), (n - i - 7/2), (i + 4/2))$	$((i + 4/2), (i + 2/2), (n - i - 5/2), (n - i - 5/2), (i + 6/2))$
$i = m - 1$	$((m + 1/2), (m - 1/2), 3, 3, (m + 3/2))$	$((n - m + 2/2), (m + 1/2), 3, 4, (m + 5/2))$
$i = m + 1$	$((m - 1/2), (m + 1/2), 2, 2, (m + 2/2))$	$((n - m/2), (m + 3/2), 1, 2, (m + 3/2))$
$i = m + 3$	$((n - i - 1/2), (n - i + 5/2), (i - m + 1/2), (i - m - 1/2), (n - i + 1/2))$	$((m - 1/2), (m + 3/2), 1, 0, (m - 1/2))$
$i = m + 5$	$((n - i - 1/2), (n - i + 5/2), (i - m + 1/2), (i - m - 1/2), (n - i + 1/2))$	$((m - 3/2), (m - 1/2), 3, 2, (m - 1/2))$
$m + 7 \leq i \leq n - 3$	$((n - i - 1/2), (n - i + 5/2), (i - m + 1/2), (i - m - 1/2), (n - i + 1/2))$	$((n - i + 1/2), (n - i + 3/2), (i - m + 3/2), (i - m + 1/2), n - i - 1)$
$i = n - 1$	$(0, 3, (m + 1/2), (m - 1/2), 1)$	$(1, 2, (m + 1/2), (m - 1/2), 0)$

Table 15: Representation of nodes in $P(n,2)$ for $n \equiv 3 \bmod 4$ and $n \geq 11$ and odd i.

i (odd)	$r(u_i\|\Re)$	$r(v_i\|\Re)$
$i = 1$	$(1, 3, (m+3/2), (m+1/2), 2)$	$(2, 2, m-3, m-4, 2)$
$3 \leq i \leq m-2$	$((i+1/2), (i+5/2), (n-i-8/2), (n-i-6/2), (i+3/2))$	$((i+3/2), (i+3/2), (n-i-6/2), (n-i-4/2), (i+5/2))$
$i = m$	$((i+1/2), (n-i/2), (n-i-8/2), (n-i-6/2), (n-i+2/2))$	$((m+3/2), (m+3/2), 2, 3, (m+5/2))$
$i = m+2$	$((i+1/2), (n-i/2), (n-i-8/2), (n-i-6/2), (n-i+2/2))$	$((m+1/2), (m+1/2), 0, 1, (m+1/2))$
$m+4 \leq i \leq m+6$	$((n-i+3/2), (n-i-1/2), (i-m/2), (i-m/2), (n-i+2/2))$	$((n-i+2/2), (n-i+2/2), i-m-2, i-m-3, (n-i+4/2))$
$m+8 \leq i \leq n-2$	$((n-i+3/2), (n-i-1/2), (i-m/2), (i-m/2), (n-i+2/2))$	$((n-i+2/2), t(n-i+2/2)n, q(i-m+2/2)h,(i-m+2/2)x, 7nC-; i-1))$
$i = n$	$(3, 0, (m+1/2), (m+3/2), 2)$	$(2, 1, (i-m+2/2), (i-m+2/2), 1)$

Lemma 3 (see [21]). *For* $n \geq 5$, $dim(P(n,2)) = 3$.

Theorem 6. *The FTMD of* $P(n,2)$, *for* $n \geq 5$ *is*

$$\wp(P(n,2)) = \begin{cases} 4, & \text{for even } n, \\ 4 \text{ or } 5, & \text{for odd } n. \end{cases} \quad (16)$$

Proof. In order to prove the theorem, following cases can be considered: □

Case 1. (When n is even)

The case is further subdivied as follows:

Case 1a. (When $n = 6$)

It can be immediately confirmed that $\mathfrak{R} = \{u_1, u_3, u_5, v_1\}$ is a FTRS for $P(6,2)$. This implies that $\wp(P(6,2)) \leq 4$. Therefore, in lights of Lemma 3.3 combined with Theorem 1, it is concluded that $\wp(P(6,2)) = 4$.

Case 1b. (When $n \equiv 0 \bmod 4$ and $n \geq 8$)

Let $n = 2m$, for even $m \geq 4$ and take $\mathfrak{R} = \{u_1, u_3, u_{m+1}, u_{m+3}\}$. Then for the vertices u_i and v_i the representations are shown in Table 8 and 9.

Case 1c. (When $n \equiv 2 \bmod 4$ and $n \geq 10$)

Let $n = 2m$, for odd $m \geq 5$ and take $\mathfrak{R} = \{u_1, u_3, u_{m+2}, u_{m+4}\}$. Then for the vertices u_i and v_i the representations are shown in Table 10 and 11.

The Cases 1b and 1c indicates that more than one zeros exist in the $AD((a,b)|\mathfrak{R})$, for each $a, b \in V(P(n,2))$ implying that $\wp(P(n,2)) \leq 4$. Therefore, in lights of Lemma 3 combined with the Theorem 1, we conclude that $\wp(P(n,2)) = 4$.

Case 2. (When $n \simeq 1 \bmod 2$)

The case is further divided as below:

Case 2a. (When $n = 5, 7$)

It can be immediately confirmed that $\mathfrak{R} = \{u_1, u_2, v_3, v_5\}$ and $\mathfrak{R} = \{u_1, u_2, v_6, v_7\}$ is FTRS for $P(5,2)$ and $P(7,2)$ respectively. This implies that $\wp(P(n,2)) \leq 4$ for $n = 5, 7$. Therefore, in lights of Lemma 3 combined with Theorem 1, it is concluded that $\wp(P(n,2)) = 4$ for $n = 5, 7$.

Case 2b. (When $n \equiv 1 \bmod 4$ and $n \geq 9$)

Let $n = 2m + 1$, for even $m \geq 4$ and take $\mathfrak{R} = \{u_{n-1}, u_n, v_{m+2}, v_{m+3}, v_n\}$, then for the vertices u_i and v_i the representations are shown in Table 12 and 13.

Case 2c. (When $n \equiv 3 \bmod 4$ and $n \geq 11$)

Let $n = 2m + 1$, for odd $m \geq 5$ and take $\mathfrak{R} = \{u_{n-1}, u_n, v_{m+2}, v_{m+3}, v_{n-1}\}$,then the representations of the vertices u_i and v_i are shown in Table 14 and 15.

The Cases 2b and 2c indicates that more than one zeros exist in the $AD((a,b)|\mathfrak{R})$, for each $a, b \in V(P(n,2))$. Hence, $\mathfrak{R}$ is a FTRS. Hence, in view of Lemma 3 combined with the Theorem 1, we conclude that $\dim(P(n,2)) = 3 < 4 \leq \wp(P(n,2)) \leq 5$. This concludes the proof. □

Table 16: Representation of nodes in $P(8,2)$ with respect to $\mathfrak{R}$.

i	$r(u_i\|\mathfrak{R})$	$r(v_i\|\mathfrak{R})$
1	(0, 1, 2, 1)	(1, 2, 3, 2)
2	(3, 3, 4, 4)	(2, 2, 3, 3)
3	(1, 0, 1, 2)	(2, 1, 2, 3)
4	(4, 3, 3, 4)	(3, 2, 2, 3)
5	(0, 1, 0, 1)	(3, 2, 1, 2)
6	(4, 4, 3, 3)	(3, 3, 2, 2)
7	(1, 2, 1, 0)	(2, 3, 2, 1)
8	(3, 4, 4, 3)	(2, 3, 3, 2)

Remark 1. This can be verified easily that $\wp(P(n,2)) = 4$, when $n = 15$, by using the resolving set $\mathfrak{R} = \{u_1, u_2, v_5, v_8\}$. Similarly, it can be verified that $\wp(P(n,2)) \neq 4$, for $n = 17$.

4. Application

The current section is included with an application of FTMD in context of navigational optimization problem is discussed. The essence of a navigational routing problem is to reach targeted location and avoiding mixing of apparently similar locations. If the targeted locations are referred as nodes and the roads connecting them as edges of a network, then the process of uniquely identifying each node of the network with respect to certain minimum collection of its reference nodes is a realization of metric dimension in networks. Further, if one of the location in the reference collection of nodes is unreachable, then the minimum number of such reference nodes providing unique identification of each location in network is realization of FTMD of network. As an illustrative case, consider navigating in a network $P(8,2)$, consisting of node set $V(P(8,2)) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ as shown in Figure 3. Then the unique identification of each node with respect to minimum collection of reference nodes $\mathfrak{R} = \{u_1, u_3, u_5, u_7\}$ is shown in Table 16.

It can be seen that the set $\mathfrak{R}$ is minimum set representing each node uniquely even if one of the nodes in it is unreachable. Therefore, it is concluded that if the navigation coordinates is in accordance with the set $\mathfrak{R}$, then the navigational routing will be optimal.

5. Conclusion

In the current study, the FTMD of the families $n-$ sunlet graph S_n and the generalized Petersen graph $P(n,t)$, for $t = 1$ is computed, which were found to be constant. We also computed the constant FTMD of $P(n,2)$ for even n and tight bounds are obtained for odd n. Finally, the article is concluded with the following open problems. [22–25].

Open Problem 1. For what values of odd n the FTMD of family of the generalized Petersen graph $P(n,2)$ is 4 or 5.

Open Problem 2. Find the FTMD of family of the generalized Petersen graphs $P(n,t)$ for $t \geq 3$.

Data Availability

All data are included within this article. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors are grateful to the reviewers for their valuable suggestions to improve the presentation of the manuscript.

References

[1] P. J. Slater, "Dominating and referece set in a graph," *Journal of Mathematical and Physical Sciences*, vol. 22, no. 4, pp. 445–455, 1988.

[2] F. Harary and R. A. Melter, "On the metric dimension of a graph," *Ars Combinatoria*, vol. 2, pp. 191–195, 1976.

[3] A. Estrado-Moreno, J. A. Rodriguez-Velaquez, and I. G. Yero, "The $k-$ metric dimension of a graph," *Applied Mathematics and Information Sciences*, vol. 9, no. 6, pp. 2829–2840, 2015.

[4] A. Estrado-Moreno, I. G. Yero, and J. A. Rodriguez-Velaquez, "$k-$Metric resolvability in graphs," *Electronic Notes in Discrete Mathematics*, vol. 46, pp. 121–128, 2014.

[5] A. Estrado-Moreno, I. G. Yero, and J. A. Rodriguez-Velaquez, "The $k-$ metric dimension of lexicographic product of graphs," *Discrete Mathematics*, vol. 339, no. 7, pp. 1924–1934, 2016.

[6] A. Estrado-Moreno, I. G. Yero, and J. A. Rodriguez-Velaquez, "The $k-$ metric dimension of corona product of graphs," *Bull. Malays. Math. Sci. Soc.*vol. 39, no. 1, pp. 135–156, 2016.

[7] A. Estrado-Moreno, I. G. Yero, and J. A. Rodriguez-Velaquez, "T"he $k-$ metric dimension of graphs: a general approach," 2016.

[8] A. Estrado-Moreno, I. G. Yero, and J. A. Rodriguez-Velaquez, "Computing the $k-$ metric dimension of graphs," *Applied Mathematics and Computation*, vol. 300, pp. 60–69, 2017.

[9] C. Hernando, M. Mora, P. J. Slater, and D. R. Wood, "fault-tolerant metric dimension of graphs," *Joint Proc. Int. Wrk.shps. Conv. in Discr. Struct. and Metric & Conv. Graph Th. No.*vol. 5, pp. 81–85, 2008.

[10] Z. Ahmad, M. O. Ahmad, A. Q. Baig, and M. Naeem, "Fault-tolerant Metric Dimension of $P(n,2)$ with Prism Graph," https://www.researchgate.net/publication/328997201_Fault-Tolerant_Metric_Dimension_of_Pn2_with_Prism_Graph.

[11] M. Basak, L. Saha, G. K. Das, and K. Tiwari, "Fault-tolerant metric dimension of circulant graphs $C_n(1,2,3)$," *Theoretical Computer Science*, vol. 817, 2019.

[12] A. Estrado-Moreno, I. G. Yero, and J. A. Rodriguez-Velaquez, "Relationship between the 2− metric dimension and the 2− adjacency dimensionin the lexicographic product of graphs," *Graphs and Combinatorics*, vol. 32, pp. 2367–2392, 2016.

[13] I. Javaid, M. Salman, M. A. Chaudhry, and S. Shokat, "Fault-Tolerance in resolvability," *Utilitas Mathematica*, vol. 80, pp. 263–275, 2009.

[14] J. B. Liu, M. Munir, I. Ali, Z. R. Hussain, and A. Ahmed, "Fault-Tolerant Metric Dimension of Wheel Related Graphs," vol. 2020, Article ID 1216542, 8 pages, 2020.

[15] H. Raza, S. Hayat, and X.-F. Pan, "On the fault-Tolerant metric dimension of convex polytopes," *Applied Mathematics and Computation*, vol. 339, pp. 172–185, 2018.

[16] A. Shabbir and T. Zumfererscu, "Fault-tolerant designs in triangular lattice network," *Discrete Mathematics*, vol. 219, pp. 179–189, 2002.

[17] C. Huo, H. Bashir, H. Zahid, and Y. Ming Chu, "On the 2-metric resolvability of graphs," *AIMS Mathematics*, vol. 5, no. 6, pp. 6609–6619, 2020.

[18] H. Bashir, Z. Zahid, A. Kashif, S. Zafar, and J.-B. Liu, "On 2-metric resolvability in rotationally-symmetric graphs," *Journal of Intelligent and Fuzzy Systems*, vol. 40, no. 6, pp. 11887–11895, 2021.

[19] H. M. Ikhlaq, H. M. A. Siddiqui, and M. Imran, "A comparative study of three resolving parameters of graphs," *Complexity*, vol. 2021, pp. 1–13, 2021.

[20] H. M. A. Siddiqui, S. Hayat, A. Khan, M. Imran, A. Razzaq, and J.-B. Liu, "Resolvability and fault-tolerant resolvability structures of convex polytopes," *Theoretical Computer Science*, vol. 796, pp. 114–128, 2019.

[21] I. Javaid, M. T. Rahim, and K. Ali, "Families of regular graphs with constant metric dimension," *Utilitas Mathematica*, vol. 75, pp. 21–33, 2008.

[22] R. Bailey and G. I. Yero, "Error-correcting codes from k-resolving sets," *Discussiones Mathematicae Graph Theory*, vol. 39, no. 2, pp. 341–355, 2019.

[23] G. Chartrand and P. Zhang, "The theory and applications of resolvability in graphs," *A Survey. Congressus Numerantium.*vol. 160, pp. 47–68, 2003.

[24] C. Poisson and P. Zhang, "The metric dimension of unicyclic graphs," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 40, pp. 17–32, 2002.

[25] H. Raza, S. Hayat, M. Imran, and X.-F. Pan, "Fault-Tolerant resolvability and extremal structures of graphs," *Mathematics*, vol. 7, no. 1, p. 78, 2019.