A CHARACTERIZATION OF CLASSES OF LINEAR TERNARY CYCLIC CODES, THEIR DESIGNS AND LATTICES

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TITLE PAGE

DECLARATION

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DEDICATION

To my dear parents, Anne and Nicholas Okombo.

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ABSTRACT

Linear cyclic ternary codes defined over the Galois field $GF(3)$ exhibit several advantages over their binary counterparts. They provide an extra option for each pulse resulting into a larger set of available codes at any given length. This thesis presents a comprehensive study of classes of linear cyclic ternary codes of length $25 \leq n \leq 50$, their associated combinatorial designs, and lattice structures. While binary codes have been extensively studied, the properties and applications of longer ternary codes remain less explored. This research aimed at addressing this gap by providing an in-depth characterization of these codes and their related mathematical structures. Using computational methods implemented in Magma software, we generated and analyzed a diverse set of linear cyclic ternary codes over $GF(3)$. The study examined key properties including minimum distance, weight distribution, and theoretical bounds for both the generated codes and their duals. We employed the Assmus-Mattson Theorem and Kramer-Mesner method to construct t-designs from these codes, revealing rich combinatorial structures. A significant contribution of this research is the exploration of the geometric properties of these codes through the construction and analysis of related lattices using Construction A. We investigated the relationships between code parameters and lattice properties, providing new insights into the structure of ternary codes from a geometric perspective. Our findings extended the existing knowledge of ternary cyclic codes, particularly for lengths exceeding 25. We identified several new codes with favorable parameters, constructed previously unreported combinatorial designs, and characterized lattices with unique properties. The results demonstrated that ternary cyclic codes exhibit high structural regularity and often produce interesting designs and lattices with properties distinct from their binary counterparts. The research revealed strong interconnections between coding theory, combinatorial design theory, and lattice theory in the context of ternary codes. We provided a multifaceted characterization framework that integrates algebraic, combinatorial, and geometric perspectives, offering a holistic understanding of these codes. This study contributes to the theoretical advancement of non-binary codes and opens new avenues for their practical applications in error correction, cryptography, and communication systems. The comprehensive analysis and novel insights presented in this thesis lay a strong foundation for future research in ternary coding theory and its intersections with combinatorics and geometry.

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LIST OF ABBREVIATIONS

BCH	:	Bose-Chaudhuri-Hocquenghem (a class of cyclic error-correcting codes)
BIBD	:	Balanced Incomplete Block Design
CRT	:	Chinese Remainder Theorem
DFT	:	Discrete Fourier Transform
GF	:	Galois Field
LDPC	:	Low-Density Parity-Check (a class of linear error-correcting codes)
MDS	:	Maximum Distance Separable
MOLS	:	Mutually Orthogonal Latin Squares
NLCS	:	Non-Linear Complementary Sequences
OEIS	:	Online Encyclopedia of Integer Sequences
PG	:	Projective Geometry
QR	:	Quadratic Residue (a type of cyclic code)
RS	:	Reed-Solomon (a class of error-correcting codes)
SDP	:	Semidefinite Programming
SNR	:	Signal-to-Noise Ratio
STS	:	Steiner Triple System
SVP	:	Shortest Vector Problem (in lattice theory)
WDCH	:	Weight Distribution of Cyclic Hamming codes
ZCC	:	Zero-Correlation Code

$\mathbb{GF}(3)$	:	The Galois field with 3 elements
n	:	Code length
k	:	Code dimension
d	:	Minimum distance of a code
$[n, k, d]$	:	Parameters of a linear code
C	:	A code (typically a linear code)
$C^{\perp}$	:	The dual code of C
$g(x)$	:	Generator polynomial of a cyclic code
$h(x)$	:	Parity-check polynomial of a cyclic code
G	:	Generator matrix of a code
H	:	Parity-check matrix of a code
$\mathrm{Aut}(C)$	:	Automorphism group of code C
$\mathrm{wt}(c)$	:	Hamming weight of a codeword c
$d(x, y)$	:	Hamming distance between vectors x and y
A_i	:	Number of codewords of weight i
$W(x, y)$	:	Weight enumerator polynomial
α	:	Primitive element of a finite field
$\mathrm{Tr}(\cdot)$	:	Trace function
χ	:	Character of a finite field
ζ	:	Primitive root of unity
λ	:	Number of blocks containing a fixed set of points in a design
t-(v, k, λ)	:	Parameters of a t-design
Λ	:	A lattice
$\det(\Lambda)$	:	Determinant of a lattice
$r(\Lambda)$	:	Covering radius of a lattice
$\gamma(\Lambda)$	:	Hermite constant of a lattice
$\theta_{\Lambda}(q)$	:	Theta series of a lattice
$\oplus$	:	Direct sum of vector spaces
$\otimes$	:	Tensor product
$\cong$	:	Isomorphic to
$\lvert S \rvert$	:	Cardinality of set S
$\lfloor x \rfloor$	:	Floor function (largest integer not greater than x)
$\lceil x \rceil$	:	Ceiling function (smallest integer not less than x)

CHAPTER ONE

INTRODUCTION

1.1 Background

In the modern digital era, the accurate and efficient transmission and storage of data is of paramount importance. However, the communication channels through which this data is transmitted, such as television, satellite, radio, and telephone, are susceptible to noise that can corrupt the message [39]. Coding theory addresses this challenge by introducing redundancy into the message to detect and correct errors that may occur during transmission [40]. The study of coding theory was pioneered by Claude Shannon in his seminal work "A Mathematical Theory of Communication" [59]. Since then, researchers have focused on developing codes that optimize the trade-off between error correction capability and efficiency [18]. Linear block codes, which exhibit a simpler structure and enable the use of matrices for encoding and decoding, have garnered significant attention [48]. Among linear block codes, cyclic codes have found extensive practical applications due to their rich algebraic structure that facilitates efficient encoding and decoding algorithms [35]. While binary cyclic codes have been widely studied [9, 33, 47, 69], the exploration of non-binary cyclic codes, particularly ternary codes, has gained traction in recent years [26, 23]. Linear cyclic ternary codes, defined over the Galois field GF(3), exhibit several advantages over their binary counterparts. They provide an extra option for each pulse, resulting in a larger set of available codes at any given length [39]. Moreover, ternary codes have found applications in various domains, such as secret sharing schemes [25], authentication codes [27], and frequency hopping sequences [30]. The study of linear cyclic ternary codes is motivated by their potential to enhance the reliability and security of modern communication systems. By understanding their algebraic structure, it becomes possible to design codes with improved error detection and correction

capabilities [32]. Furthermore, the construction of combinatorial designs [64] and lattices [17] from these codes offers new avenues for their application and analysis. This thesis aims to contribute to the growing body of knowledge on linear cyclic ternary codes by exploring their properties, constructing associated designs and lattices, and characterizing them based on these findings. By leveraging on the rich algebraic structure of these codes, we seek to develop new insights that can inform the design of more efficient and resilient communication systems.In the modern digital era, the accurate and efficient transmission and storage of data is of paramount importance. However, the communication channels through which this data is transmitted, such as television, satellite, radio, and telephone, are susceptible to noise that can corrupt the message [39]. Coding theory addresses this challenge by introducing redundancy into the message to detect and correct errors that may occur during transmission [40]. The study of coding theory was pioneered by Claude Shannon in his seminal work "A Mathematical Theory of Communication" [59]. Since then, researchers have focused on developing codes that optimize the trade-off between error correction capability and efficiency [18]. Linear block codes, which exhibit a simpler structure and enable the use of matrices for encoding and decoding, have garnered significant attention [48]. Among linear block codes, cyclic codes have found extensive practical applications due to their rich algebraic structure that facilitates efficient encoding and decoding algorithms [35]. While binary cyclic codes have been widely studied [9, 33, 47, 69], the exploration of non-binary cyclic codes, particularly ternary codes, has gained traction in recent years [26, 23]. Linear cyclic ternary codes, defined over the Galois field GF(3), exhibit several advantages over their binary counterparts. For instance, they provide an extra option for each pulse, resulting in a larger set of available codes at any given length [39]. Moreover, ternary codes have found applications in various domains, such as secret sharing schemes [25], authentication codes [27], and frequency hopping sequences [30]. The study of linear cyclic ternary codes is motivated by their potential to enhance the reliability and security of mod-

ern communication systems. By understanding their algebraic structure, it becomes possible to design codes with improved error detection and correction capabilities [32]. Furthermore, the construction of combinatorial designs [64] and lattices [17] from these codes offers new avenues for their application and analysis. This thesis aims to contribute to the growing body of knowledge on linear cyclic ternary codes by exploring their properties, constructing associated designs and lattices, and characterizing them based on these findings. By leveraging on the rich algebraic structure of these codes, we seek to develop new insights that can inform the design of more efficient and resilient communication systems.

1.2 Definition of Significant Terms

In this section,we give basic definitions and examples that are fundamental in this study.

Definition 1.2.1. ***Code Characterization***

In the context of Coding Theory, characterization refers to the process of describing and analyzing the fundamental properties and features that uniquely define a particular class of codes.

Definition 1.2.2. ***Linear Code***

A code C is linear if the sum of any two codewords in C is also a codeword in C. In other words, C forms a vector subspace over its field of definition.

Definition 1.2.3. ***Ternary Code***

Relating to a base-3 number system. In coding theory, ternary codes use an alphabet of three symbols, typically $0, 1, 2$, which correspond to elements of the Galois field $GF(3)$.

Definition 1.2.4. ***Cyclic Code***

A linear code C is cyclic if any cyclic shift of a codeword in C is also a codeword in C. Mathematically, if $(c_0, c_1, \ldots, c_{n-1})$ is in C, then $(c_{n-1}, c_0, \ldots, c_{n-2})$ is also in C.

Definition 1.2.5. ***Design***

In combinatorial mathematics, a design is a set of points with a family of subsets (called blocks) that satisfy certain regularity conditions. In the context of Coding Theory, designs often arise from the supports of codewords of a given weight

Definition 1.2.6. ***Lattice***

In mathematics, a lattice is a discrete subgroup of $\mathbb{R}^n$*. In the context of coding theory, lattices are often constructed from linear codes using methods such as Construction A, where codewords are mapped to points in Euclidean space. These code-based lattices inherit properties from their underlying codes and provide a geometric perspective on code structure and performance.*

Definition 1.2.7. *A field is a commutative ring with identity* $1 \neq 0$ *in which every nonzero element is a unit. The field is finite if* $\mathbb{F}$ *has a finite number of elements. A field with* q *elements is denoted* $\mathbb{F}_q$ *or* $GF(q)$ *read the Galois field with* q *elements. In particular,* $\mathbb{F}_2$ *or* $GF(2)$ *is the two element field called the binary field and the two elements are denoted 0 and 1 while field with three elements is denoted* $\mathbb{F}_3$ *or* $GF(3)$ *is called the ternary field and the three elements are listed as 0, 1 and 2.*

Definition 1.2.8. *A block code* C *is a set of* M *codewords*

$$C = c_1, c_2, \cdots, c_M$$

$$c_i = (c_{i0}, c_{i1}, \cdots, c_{in-1})$$

where the codewords are n*-tuples and* n *is the length of the code. The elements* c *belong to a finite alphabet of* q *symbols.*

Definition 1.2.9. *Let* $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2 \ldots, y_n)$ *be vectors of the same length. The Hamming distance* $d_H(x, y)$ *between them is the total number of positions in which corresponding coordinates differ. In symbols,*

$$d_H(x, y) = \sum_{i=1}^{n} x_i \neq y_i$$

Definition 1.2.10. *The minimum distance d of a code C is the minimum Hamming distance between any pair of codewords.*

Definition 1.2.11. *An (n, k) block code C is a k- dimensional subspace of the vector space C.*

Definition 1.2.12. *A generator matrix G of an (n, k) code C is a $k \times n$ matrix whose rows are linearly independent.*

Definition 1.2.13. *A parity check is a vector h of length n which satisfies*

$$Gh^T = 0$$

where h^T is the transpose of h and G, the generator matrix of C.

Definition 1.2.14. *A parity check matrix H for an (n, k) code C is an $(n - k) \times n$ matrix whose rows are linearly independent checks.*

Definition 1.2.15. *The code spanned by the rows of H is called the dual code$C^\perp$* $C^\perp = \{x \in H \mid x \cdot c = 0, \forall c \in G\}$

Definition 1.2.16. *The minimum distance of a code d is the minimum Hamming distance between any pair of codewords.*

Definition 1.2.17. *The weight distribution of an (n, k) code C is a vector $A = (A_0, A_1, A_3, \cdots, A_n)$ represented a polynomial $A(z)$ where A_i is the number of codewords of weight w,*

$$A(z) = A_0 + A_1 z + A_2 z_2 + \ldots + A_n z_n.$$

Definition 1.2.18. *The Hamming weight of a vector v is defined to be the number of its nonzero coordinates.*

Definition 1.2.19. *The support of a vector v(codeword, in our case) is the set of coordinates with non-zero values that is, $\{1 \leq i \leq n : v_i \neq 0\}$*

Definition 1.2.20. *A code is called perfect if*

$$\frac{q^n}{M} = V_q(n,t)$$

Definition 1.2.21. *A structure (or an incidence structure) is two finite sets of objects called points and blocks with an incidence relation between them.*

Definition 1.2.22. *Let F be a field. An order n square matrix M over F in which each row and each column has just one non-zero element of F is called a monomial matrix of order n.*

Definition 1.2.23. *An incidence structure $\mathcal{D}$ consisting of a block set $\mathcal{B}$ on a point set $\mathcal{P}$ of size v is a $t-(v,k,\lambda)$ design or $t-$ design for short if every block is incident with precisely k points and any set of t distinct points are together incident with precisely λ blocks.*

Definition 1.2.24. *A design is called symmetric if it has the same number of points and blocks.*

Definition 1.2.25. *An automorphism of a design $\mathcal{D}$ is a permutation on P which sends blocks to blocks.*

Definition 1.2.26. *The set of monomials on a code C form a group called the monomial automorphism of C.*

Definition 1.2.27. *Let M_γ be the set of maps where M is a monomial matrix and γ an automorphism on a field that map a code C to itself. The set of all M_γ is called the automorphism group of the code C.*

Definition 1.2.28. *Let D be a diagonal matrix, P a permutation matrix and γ an automorphism of GF(q). An automorphism group is called t-transitive if there is an element DP_γ of the automorphism group for each pair of t-element ordered sets of coordinates such that $Pesndsthefirstsettothesecond.$*

Definition 1.2.29. *For a structure $\mathcal{D}$ with $\mid \mathcal{P} \mid$ points and $\mid \mathcal{B} \mid$ blocks where $\mid \mathcal{P} \mid$ ¿ 0 and $\mid \mathcal{B} \mid$ ¿ 0, the incidence matrix of $\mathcal{D}$ is the $\mid \mathcal{P} \mid$ by $\mid \mathcal{B} \mid$ matrix of 0s and 1s where the entry is indexed by $(\mathcal{B}, x)$ is 1 if and only if the block B is incident with the point x.*

Definition 1.2.30. *Given n linearly independent vectors $b_1, b_2, \cdots, b_n \in \mathbb{R}^n$, the lattice generated by them is defined as*

$$\mathcal{L}(b_1, b_2, \cdots, b_n) = \{\sum x_i b_i \mid x_i \in \mathbb{Z}^n\}.$$

The vectors $b_1, b_2, \cdots, b_n$ are defined as the basis of the lattice. A For the $m \times n$ matrix B whose columns are $b_1, b_2, \cdots, b_n$ the lattice generated by B is

$$\mathcal{L}(B) = \mathcal{L}(b_1, b_2, \cdots, b_n) = \{Bx \mid x \in \mathbb{Z}^n\}.$$

The rank of the lattice $\mathcal{L}$ as the n columns of the matrix B. If the matrix B has the dimension m and $n = m$ then the lattice is called a full rank lattice.

Definition 1.2.31. *The span of lattice $\mathcal{L}$ is the linear space spanned by its vectors.*

$$span(\mathcal{L}(B)) = span(B) = \{By \mid y \in \mathbb{R}^n\}$$

Definition 1.2.32. *A generator matrix for the lattice L is one whose columns are the basis for L.*

Definition 1.2.33. *The fundamental parallelepiped for any lattice of basis B is defined as*

$$\mathcal{P}(B) = \{Bx \mid x \in [0, 1)^n\}$$

which is, essentially, the space spanned by its basis when restricting the scalar of the basis to the domain [0, 1).

Definition 1.2.34. *A matrix U in $\mathbb{Z}^{n \times n}$ is called unimodular if $detU = 1$.*

Definition 1.2.35. *A lattice $L \in \mathbb{R}^n$ is said to cover $\mathbb{R}^n$ with $\mathcal{B}_r$ provided*

$$\bigcup_{\lambda \in L} (\lambda + \mathcal{B}_r) = \mathbb{R}^n$$

Definition 1.2.36. *A lattice is said to pack $\mathcal{B}_r$ if*

$$\lambda_1, \lambda_2 \in L, \lambda_1 \neq \lambda_2 \Rightarrow (\lambda_1 + \mathcal{B}_r) \cap (\lambda_2 + \mathcal{B}_r) = \varnothing$$

Definition 1.2.37. *The covering radius of the lattice L is*

$$r_{cov}(L) = min\{r \mid L \; covers \; \mathbb{R}^n \; with \; \mathcal{B}_r\}$$

Definition 1.2.38. *The packing radius of L is defined as*

$$r_{pack}(L) \; = sup \; \{ \; r \mid L \; packs \; \mathcal{B}_r\}$$

Definition 1.2.39. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ be an n-dimensional vector with integer entries and q, a specific integer. The modulo-q-reduction of the vector $\mathbf{x}$ is*

$$\mathbf{x} \; mod \; p = (x_1 mod \; q, x_2 \; mod \; q, \; \ldots, \; x_n \; mod \; q),$$

where $s \; mod \; p = r$ if $s = pq + r$ and $0 \leq r \leq q$.

Definition 1.2.40. *A linear $[n, k]$ code C lifted over $\mathbb{F}_q^n$ to a lattice*

$$L_C = \{\mathbf{x} \in \mathbb{Z}^n : x \; mod \; q \in C\} \; or \; q\text{-}ary \; lattices.$$

called a modulo-p-lattice.

1.3 Statement of the Problem

While significant progress has been made in the study of binary cyclic codes, the exploration of linear cyclic ternary codes remains limited. The rich algebraic structure of these codes, defined over the Galois field $GF(3)$, presents opportunities for improving error detection and correction in modern communication systems. However, the characterization of these codes based on their properties, associated combinatorial designs, and lattices has not been fully explored. This thesis aims to address this gap in knowledge by conducting a comprehensive analysis of linear cyclic ternary codes and their related structures.

1.4 General Objective

The main objective of this study is to conduct a comprehensive characterization and analysis of linear cyclic ternary codes of length $25 \leq n \leq 50$ over the Galois field $GF(3)$.

1.5 Specific Objectives

The specific objectives of this research are as follows:

i. To generate a diverse set of linear cyclic ternary codes of length $n : 25 \leq n \leq 50$ over $GF(3)$ and determine the key algebraic properties, including minimum distance, weight enumerators, and weight distribution.

ii. To construct combinatorial designs from the generated codes using techniques based on the codes' automorphism groups and characterize the designs.

iii. To determine the relationship between linear cyclic ternary codes and lattices by constructing lattices from the generated codes and analyzing their properties.

iv. To compare the findings with existing results in the literature and discuss the potential implications for enhancing error detection and correction in communication systems.

1.6 Significance of the Study

By achieving these objectives, this research aims to contribute to the theoretical understanding of linear cyclic ternary codes and their related structures. The insights gained from this study may have practical applications in the design of more efficient and robust coding schemes for various communication scenarios. Furthermore, the characterization of these codes based on their properties, designs, and lattices can open up new avenues for future research in coding theory and its applications.

1.7 The Scope of the Study

1. The study is limited to linear cyclic ternary codes and does not consider other types of codes, such as non-linear or non-cyclic codes.

2. The research focuses on codes over the Galois field $GF(3)$ and does not extend to other finite fields.

3. The generation of codes is limited to lengths $n : 25 \leq n \leq 50$ due to computational constraints and the complexity of analyzing longer codes.

4. The construction of combinatorial designs is restricted to techniques based on the automorphism groups of the codes, and other design construction methods are not explored.

5. The study of lattices is limited to those constructed directly from the generated linear cyclic ternary codes and does not consider other lattice construction techniques.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction to Coding Theory

Coding Theory is a branch of mathematics and computer science that deals with the design and analysis of codes for the efficient and reliable transmission of information over noisy communication channels [40]. The primary goal of coding theory is to introduce redundancy into the transmitted message in a structured manner, enabling the receiver to detect and correct errors that may have occurred during transmission [59].

The origins of coding theory can be traced back to the pioneering work of Claude Shannon in 1948 [59]. Shannon's paper, "A Mathematical Theory of Communication," laid the foundation for information theory and introduced the concept of channel capacity, which represents the maximum rate at which information can be reliably transmitted over a noisy channel [12].

In coding theory, a code is a set of rules that maps the original message (information) to a codeword, which is then transmitted over the communication channel [48]. The codeword is typically longer than the original message, as it includes redundant information that enables error detection and correction. The process of adding redundancy to the message is called encoding, while the process of recovering the original message from the received codeword is called decoding [43].

Codes are classified into various categories based on their properties and structure. Some common types of codes include:

1. Linear codes: These codes form a linear subspace over a finite field, enabling the use of algebraic tools for their analysis and design [48].

2. Cyclic codes: A subclass of linear codes, cyclic codes have the property that any cyclic shift of a codeword is also a codeword [35].

3. Block codes: These codes operate on fixed-size blocks of information and produce fixed-size codewords [43].

4. Convolutional codes: Unlike block codes, convolutional codes operate on a continuous stream of information and generate codewords based on the current input and a fixed number of previous inputs [43].

5. Algebraic-geometric codes: These codes are constructed using tools from algebraic geometry and have been shown to achieve good performance in certain settings [70].

The design of codes suited to different applications involves finding a balance between the following key parameters [18, 60]:

1. Code length, n: The number of symbols or digits in a codeword

2. Number of codewords, M: The number of codewords in a code of length n.

3. the distance between pairs of codewords in a code, d

4. Code rate: The ratio of the number of information symbols to the total number of symbols in a codeword.

5. Error-correcting capability: The maximum number of errors that the code can correct.

6. Complexity: The computational resources required for encoding and decoding.

These have been analyzed in this thesis for linear ternary cyclic codes.

Some well-known examples of codes include Hamming codes, Reed-Solomon codes, Golay codes, and Low-Density Parity-Check (LDPC) codes [43, 50]. These codes have found applications in various domains, such as data storage, satellite communication, mobile communication, and error correction in digital systems [45].

The study of coding theory has led to the development of various bounds and limitations on the performance of codes. Some notable bounds include the Hamming

bound, the Singleton bound, and the Gilbert-Varshamov bound [54]. These bounds provide theoretical limits on the achievable error-correcting capability of codes and serve as benchmarks for evaluating the performance of practical coding schemes. The bounds satisfied by linear ternary cyclic codes is determined in this work.

With many applications in internet, mobile phones, and streaming services technologies, coding theory is key in this digital era. It is this that informed the need to undertake a characterization of linear cyclic ternary codes by their algebraic properties and those of the t-designs and lattices constructed from them. The suitability of a code for error correction depends on its properties, while lattices from codes find applications in secure encryption methods and in improving the efficiency wireless communication while t-designs from codes are useful in building codes that are efficient for data storage and transmission.

In summary, coding theory provides a mathematical framework for the design and analysis of codes that enable reliable communication over noisy channels. The concepts and techniques developed in coding theory have found widespread applications in modern communication systems, making it an essential area of study for researchers and practitioners in the field of information theory and communications.

2.2 Linear Block Codes

Linear block codes are a fundamental class of codes in coding theory that exhibit a linear structure, enabling the use of algebraic tools for their analysis and design [48]. These codes operate on fixed-size blocks of information symbols and produce fixed-size codewords. The linearity property of these codes simplifies the encoding and decoding processes, making them attractive for practical implementations [43].

That linearity of these codes ensures that any linear combination of valid codewords is also a valid codeword. This property enhances error detection and correction both during data transmission and storage[10]. Linear block codes are generally described by generator matrix and parity-check matrix and these define the encoding

and decoding processes [61]. In systematic linear block codes the actual message is included in the codeword (besides the redundancy symbols), again this facilitates error correction [49] In advanced designs that include reversible logic gates, linear block codes can detect and correct multiple errors [61].

Though linear block codes are useful and are widely used in communication systems in which data integrity is important [10], they are not optimal in some situations such as in high speed applications [49]. Linear cyclic block codes, simply called cyclic codes, offer more structure and properties.

In summary, linear block codes are a class of codes that exhibit a linear structure and operate on fixed-size blocks of information symbols. The key parameters of a linear block code include its length, dimension, minimum distance, and weight distribution. These parameters determine the error-correcting capability and performance of the code. The study of linear block codes has led to the development of various code construction techniques, such as Reed-Solomon codes, Golay codes, and Hamming codes, which have found widespread applications in modern communication systems.

2.3 Cyclic Codes

Cyclic codes are a subclass of linear block codes that possess a unique algebraic structure, making them particularly suitable for efficient encoding and decoding [35]. The defining property of cyclic codes is that any cyclic shift of a codeword is also a codeword, a property that allows their implementation using shift register circuits which are advantageous in communication systems and data storage. Besides being expressed by use of generator matrices and parity-check matrices, linear cyclic codes are also defined by generator polynomials over finite fields $GF(q)$), especially q prime. In this work $q = 3$. The same codes have another algebraic structure; as cyclic multiplicative groups which allows for the generation of cyclic codes from polynomial rings, a property that allows for varying the code length [5]. Moreover linear cyclic

codes have efficient encoding and decoding algorithms which are more effective than other linear block codes [74].

Cyclic linear codes have the advantage that longer codes can be generated from smaller polynomial rings, a property that enhances their practicability [5]. Besides this cyclic codes attain optimal or near-optimal bounds on linear codes, a property crucial for error correction [24]. The mathematical structure of cyclic linear codes has has found them various practical applications in consumer electronics, data storage systems, secret sharing schemes and other communication systems [24, 74]. One limitation of cyclic linear codes is that determining optimal generator polynomials for specific applications remains a challenge and hence the need to continue research in the structure and properties of classes of cyclic linear codes[24, 55].

Computing is preliminarily binary but lately, ternary computing is gaining interest. In fact their extensive use in practical applications have seen cyclic codes studied extensively and they remain an area of research interest. The weights and weight enumerators of binary cyclic codes have been studied quite extensively [4, 6, 8, 16, 47, 56], their automorphism groups [2, 7], their combinatorial designs [67, 48, 20] and even lattices [17, 38, 53]. This is not surprising since both computer systems and communication channels frequently use binary coding systems. With interest growing interest ternary coding such as the Ternary Research Group at the University of South-Eastern Norway there is need to understand ternary codes better and more especially cyclic linear ones [11]. Cyclic codes can be represented using polynomial notation, where each codeword $(c_0, c_1, \ldots, c_{n-1})$ is associated with a polynomial $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$.

Theorem 2.3.1 (Polynomial Representation [68]). *A linear block code C of length n over a finite field F is a cyclic code if and only if C is an ideal in the quotient ring $F[x]/(x^n - 1)$.*

This theorem establishes a connection between cyclic codes and polynomial rings, enabling the use of algebraic tools for their study.

Theorem 2.3.2 (Ideal Structure [68]). *A cyclic code C of length n over a finite field F is an ideal in $F[x]/(x^n - 1)$ generated by its generator polynomial $g(x)$.*

The generator polynomial $g(x)$ of a cyclic code C divides $x^n - 1$, and the dimension of C is given by $k = n - \deg(g(x))$.

Theorem 2.3.3 (Encoding [68]). *Given a generator polynomial $g(x)$ of a cyclic code C and an information polynomial $u(x)$ of degree less than k, the corresponding codeword polynomial $v(x)$ is obtained by the polynomial multiplication: $v(x) = u(x)g(x) \mod (x^n - 1)$.*

The encoding process for cyclic codes can be efficiently implemented using shift registers, making them attractive for hardware implementations.

Lemma 2.3.1 (Parity-Check Property [35]). *A polynomial $v(x)$ is a codeword of a cyclic code C with parity-check polynomial $h(x)$ if and only if $v(x)h(x) = 0 \mod (x^n - 1)$.*

The parity-check polynomial $h(x)$ plays a role similar to the parity-check matrix in linear block codes and can be used for error detection and syndrome computation.

Theorem 2.3.4 (BCH Bound [43]). *Let C be a cyclic code of length n over a finite field F with generator polynomial $g(x)$. If $g(x)$ has t consecutive roots $\alpha^i, \alpha^{i+1}, \ldots, \alpha^{i+t-1}$, where α is a primitive n-th root of unity in an extension field of F, then the minimum distance of C is at least $t + 1$.*

The BCH bound provides a lower bound on the minimum distance of a cyclic code based on the number of consecutive roots of its generator polynomial. This bound is used in the construction of BCH codes, which are a well-known class of cyclic codes with good error-correcting capabilities.

Cyclic codes have several advantages over general linear block codes:

1. Efficient encoding and decoding algorithms based on shift registers and polynomial operations.

2. Compact representation using generator and parity-check polynomials.

3. Ability to construct codes with good error-correcting properties, such as BCH and Reed-Solomon codes.

4. Suitability for hardware implementations due to their cyclic structure.

In summary, cyclic codes are a subclass of linear block codes that exhibit a cyclic shift property. They can be represented using polynomial notation and possess an algebraic structure that enables efficient encoding and decoding. The study of cyclic codes has led to the development of important classes of codes, such as BCH and Reed-Solomon codes, which have found widespread applications in error correction and data storage systems.

2.4 Linear Cyclic Ternary Codes

Cyclic codes have greater efficiency and better error correction than other linear block codes but their performance depends upon the chosen generator polynomial and the nature of of application in question [55].

Currently there is growing interest in ternary computing and ternary codes this work being one of them. HaBITS [42] is an effort to develop an efficient and reversible translation between binary and ternary computing systems. The University of South-Eastern Norway has established the Ternary Research Group with the mission to "to advance the technological readiness level and push ternary and mixed-radix (binary and ternary) computing on the global agenda ". There are distinct advantages of linear cyclic ternary codes over their binary counterparts that extend extend across algebraic, combinatorial and geometric aspects which in turn can influence code performance, design versatility and applications in communication and cryptographic systems.

Linear ternary cyclic codes are a specialized class of linear codes that exhibit unique properties and applications in communication and storage systems. These

codes are characterized by their cyclic nature which allows for efficient encoding and decoding processes. Recent research has focused on constructing of infinite families of these codes, particularly with specific lengths and dimensions while ensuring minimum distances.

One frequently used method for constructing linear cyclic codes is by the generator polynomial which is actually a factor of

Ternary codes are a class of error-correcting codes defined over the finite field GF(3), which consists of the elements $\{0, 1, 2\}$. Compared to binary codes, ternary codes offer an additional symbol, allowing for more efficient encoding and a larger set of available codewords [39]. Linear cyclic ternary codes, in particular, have garnered interest due to their algebraic structure and potential applications in various domains, such as secret sharing schemes [25], authentication codes [27], and frequency hopping sequences [30].

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Linear ternary codes are a subclass of ternary codes that form a linear subspace over GF(3) as stated in the following definition; The generator matrix and parity-check matrix of a linear ternary code are defined similarly to those of binary linear codes, with the operations performed over GF(3) as given in the theorem below;

Theorem 2.4.1 (Ternary Cyclic Code [35])**.** *A linear ternary code C of length n is a cyclic code if and only if C is an ideal in the quotient ring* $\mathrm{GF}(3)[x]/(x^n - 1)$.

This theorem 2.4.1 establishes the connection between ternary cyclic codes and polynomial rings over GF(3), enabling the use of algebraic tools for their analysis and

design.

Previous work on linear cyclic ternary codes has explored various aspects, including their construction, minimum distance, and weight distribution as given below:

Lemma 2.4.1 (Irreducible Cyclic Ternary Codes [46]). *An irreducible cyclic ternary code of length n and dimension k exists if and only if k divides n and $3^k - 1$ divides $3^n - 1$.*

The lemma 2.4.1 provides a necessary and sufficient condition for the existence of irreducible cyclic ternary codes.

Linear cyclic ternary codes satisfy the familiar bounds on minimum distances as stated in the proposition 2.4.1 below.

Proposition 2.4.1 (Minimum Distance Bounds [66]). *Let C be a linear cyclic ternary code of length n and dimension k. Then, the minimum distance d of C satisfies:*

1. $d \leq n - k + 1$ *(Singleton bound)*

2. $d \leq 3\lfloor (n-1)/3 \rfloor$ *(Plotkin bound)*

These bounds provide upper limits on the minimum distance of linear cyclic ternary codes and are useful in assessing their error-correcting capabilities.

Theorem 2.4.2 (Perfect Ternary Golay Code [43]). *The ternary Golay code is a perfect linear cyclic code with parameters $(11, 6, 5)$ over $\mathrm{GF}(3)$. It is the unique perfect ternary code with these parameters.*

The ternary Golay code is a well-known example of a perfect ternary code, which achieves the sphere-packing bound with equality.

Several researchers have investigated the weight distribution of linear cyclic ternary codes.

Lemma 2.4.2 (Weight Distribution of Irreducible Cyclic Ternary Codes [52]). *Let C be an irreducible cyclic ternary code of length n and dimension k. Then, the weight*

distribution of C is given by:

$$A_i = \begin{cases} (3^k - 1)/2 & \textit{for } i = (3^k - 1)/2, \\ 1 & \textit{for } i = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

This lemma provides the weight distribution for a specific class of irreducible cyclic ternary codes.

Recent work on linear cyclic ternary codes includes the construction of optimal codes with specific parameters [24], the study of their duals [23], and the exploration of their applications in various domains [25, 27, 30].

In summary, ternary codes are error-correcting codes defined over the finite field GF(3). Linear cyclic ternary codes are a subclass of ternary codes that exhibit a cyclic structure and can be represented using polynomial rings over GF(3). Previous work on linear cyclic ternary codes has explored their construction, minimum distance bounds, weight distribution, and applications for certain lengths not exceeding 25. The study of ternary codes and their properties continues to be an active area of research in coding theory.

2.5 Weight Enumerators and Weight Distributions

The weight distribution of a code provides insights into its error-correcting performance and is often used in the analysis and design of codes. Let C be a linear block code with weight distribution $(A_0, A_1, \ldots, A_n)$ and $C^\perp$ be its dual code with weight distribution $(A_0^\perp, A_1^\perp, \ldots, A_n^\perp)$, then the weight distributions of C and $C^\perp$ are related by the MacWilliams identity:

$$A_j^\perp = (1/|C|) \sum_{i=0}^{n} K_j(i) A_i,$$

where $K_j(i)$ is the Krawtchouk polynomial of degree j [51]. The MacWilliams identity establishes a connection between the weight distributions of a code and its dual, providing a powerful tool for the analysis of linear block codes.

Weight enumerators and weight distributions are important concepts in coding

theory that provide valuable information about the structure and properties of error-correcting codes. They play a crucial role in the analysis of code performance, the estimation of error probabilities, and the design of efficient decoding algorithms [32]. The weight enumerator provides a compact representation of the weight distribution of a code. The weight distribution provides a more detailed description of the weight structure of a code compared to the weight enumerator.

Theorem 2.5.1 (MacWilliams Identity [51]). *Let C be a linear code of length n over a finite field $\mathbb{F}_q$, and let $C^\perp$ be its dual code. The weight enumerators of C and $C^\perp$ are related by the MacWilliams identity:*

$$W_C^\perp(x, y) = (1/|C|)W_C(x + (q-1)y, x - y).$$

The MacWilliams identity establishes a connection between the weight enumerators of a code and its dual, providing a powerful tool for the analysis of linear codes.

Theorem 2.5.2 (Average Weight [43]). *The average weight of a linear code C of length n over a finite field $\mathbb{F}_q$ is given by:*

$$w_{avg} = n(q-1)/q.$$

This Theorem provides a simple formula for computing the average weight of a linear code based on its length and the size of the underlying field.

Lemma 2.5.1 (Pless Power Moments [43]). *Let C be a linear code of length n over a finite field $\mathbb{F}_q$ with weight distribution $(A_0, A_1, \ldots, A_n)$. The Pless power moments S_i are defined as:*

$$S_i = \sum_{j=0}^{n} j^i A_j.$$

The Pless power moments satisfy a set of equations that relate them to the code parameters and the MacWilliams identity.

The Pless power moments provide additional constraints on the weight distribution of a code and are useful in the computation of weight enumerators and the

classification of codes.

The weight distribution of a code has significant relevance in various aspects of coding theory:

1. Error Correction Capability: The minimum distance of a code, which determines its error-correcting capability, is closely related to its weight distribution. Codes with a larger minimum distance have a better ability to correct errors.

2. Error Probability Estimation: The weight distribution can be used to estimate the probability of error for a given decoding algorithm. This information is crucial in assessing the performance of a code and designing efficient decoding schemes.

3. Code Classification: Weight enumerators and distributions are used as invariants for classifying and distinguishing different codes. Codes with the same weight enumerator are considered equivalent under certain conditions.

4. Code Design: Knowledge of the weight distribution can guide the design of codes with desirable properties, such as balanced weight distributions or optimal minimum distances.

Recent research on weight enumerators and distributions of cyclic codes includes the study of binary cyclic codes with few weights [29], the computation of weight distributions for specific classes of cyclic codes [71], and the investigation of the relationship between weight distributions and code automorphisms [2].

In summary, weight enumerators and weight distributions are fundamental concepts in coding theory that provide important information about the structure and properties of error-correcting codes. They are essential tools for the analysis, classification, and design of codes, and they play a crucial role in estimating error probabilities and assessing code performance. The study of weight enumerators and distributions continues to be an active area of research, with ongoing efforts to characterize

the weight structures of various classes of codes and explore their applications in communication systems.

2.6 Designs and Codes

Combinatorial designs and error-correcting codes are two closely related areas of study that have numerous connections and applications. Combinatorial designs, such as block designs and t-designs, can be used to construct codes with desirable properties, while codes can be used to derive new designs with specific parameters [70]. While in this work the study t-designs are constructed from linear cyclic ternary codes, construction of codes from t-designs is better known.

Block designs are often denoted as (v, k, λ)-designs, where v is the number of points, k is the size of each block, and λ is the number of blocks containing any pair of distinct points. The generalization of block designs is t-designs, where the parameter t specifies the size of the subsets being considered. As stated in the theorem 2.6.1 below, the incidence matrix of a t-design can generate a linear code.

Theorem 2.6.1 (Incidence Matrix [3])**.** *Let $\mathcal{D}$ be a block design $(V, \mathcal{B})$ with $|V| = v$ and $|\mathcal{B}| = b$. The incidence matrix of $\mathcal{D}$ is a $v \times b$ matrix $M = (m_{ij})$, where $m_{ij} = 1$ if the i-th point is contained in the j-th block, and $m_{ij} = 0$ otherwise.*

The reverse process of constructing t-designs from linear codes has been less studied, and that is what is investigated in this work for linear cyclic ternary codes. Recent research on construction of t-designs from linear codes has focused on finding infinite families of codes that yield 2-,3- and even 4-designs [21, 28, 36, 63, 72, 73] mostly on binary codes with fewer such as [73] mentioning a family of ternary codes and [28] a ternary Golay code. Thus there is still room for further research on linear cyclic ternary codes holding various t-designs.

There are three general methods used to construct t-designs from linear codes namely the Assmus-Mattson Theorem and the Kramer-Mesner method and the use of incidence matrices. The use of incidence matrices is indirect and is given by the

theorem immediately below is not used in this thesis because it is effective for very small block sizes only [22].

Theorem 2.6.2 (Incidence Matrix [3]). *Let $\mathcal{D}$ be a block design $(V, \mathcal{B})$ with $|V| = v$ and $|\mathcal{B}| = b$. The incidence matrix of $\mathcal{D}$ is a $v \times b$ matrix $M = (m_{ij})$, where $m_{ij} = 1$ if the i-th point is contained in the j-th block, and $m_{ij} = 0$ otherwise.*

Lemma 2.6.1 (Assmus-Mattson Theorem [43]). *Let C be a linear code over $\mathbb{F}_q$ with minimum distance d, and let $C^\perp$ be its dual code with minimum distance $d^\perp$. If there exists an integer $t \geq 1$ such that:*

1. *$d > (t+1)q^{t-1}$, and*

2. *$d^\perp > (t+1)q^{t-1}$,*

then the supports of the codewords of any fixed weight in C form a t-design.

The Assmus-Mattson Theorem provides a powerful connection between codes and designs and is the most prominent method used for the construction of t-designs from linear codes. Either of two conditions must be satisfied by the supports of codewords of fixed weight d of an $[n, k, d]$ code C over the field $\mathbb{F}_q$ ($q = 3$ in this thesis), whose dual code $C^\perp$ has codewords of fixed weight $d^\perp$:

1. The weights of C within the set $1, 2, \ldots, n - t$ must not exceed $d^\perp - t$ or

2. The weights of $C^\perp$ within the set $1, 2, \ldots, n - t$ must not exceed $d - t$

The Kramer-Mesner method employs linear codes to construct t-designs by leveraging the properties of incidence matrices and the regularity of these codes. This approach connects coding theory with combinatorial designs, allowing for the systematic generation of t-designs through specific classes of linear codes. The following description outlines the key mathematical properties and methodologies involved in this construction.

Recent research on the connections between designs and codes includes the construction of t-designs from linear codes using their automorphism groups [31], the

derivation of new codes from combinatorial designs [62], and the study of the relationship between the weight distributions of codes and the parameters of the corresponding designs [64].

The connections between designs and codes have numerous applications, including:

1. Code Construction: Combinatorial designs can be used to construct codes with desired properties, such as high minimum distance or specific weight distributions.

2. Design Construction: Codes can be used to derive new combinatorial designs with specific parameters, leading to the discovery of previously unknown designs.

3. Cryptography: The properties of designs and codes make them suitable for use in various cryptographic schemes, such as secret sharing and authentication codes [25, 27].

4. Quantum Error Correction: Combinatorial designs have been used to construct quantum error-correcting codes, which are essential for reliable quantum communication and computation [70].

In summary, combinatorial designs and error-correcting codes are intimately connected, with each area providing tools and insights for the other. The incidence matrices of designs can be used to construct linear codes, while the distance properties and weight enumerators of codes can be used to derive t-designs. The study of the connections between designs and codes continues to be an active area of research, with ongoing efforts to explore new construction methods, characterize the properties of the resulting objects, and find novel applications in various domains.

2.7 Lattices and Codes

Lattices and codes are two fundamental objects in discrete mathematics and have numerous connections and applications in various fields, including cryptography, coding theory, and communication systems. Code-based lattice constructions provide a way to obtain lattices with desirable properties from error-correcting codes [17].

Definition 2.7.1 (Lattice [17]). *A lattice Λ is a discrete subgroup of $\mathbb{R}^n$, where $\mathbb{R}$ is the set of real numbers and n is a positive integer. Equivalently, a lattice is the set of all integer linear combinations of a set of linearly independent vectors $\{b_1, b_2, \ldots, b_k\}$ in $\mathbb{R}^n$, called a basis of the lattice.*

Lattices can be represented as the set of points $\Lambda = \{\sum_{i=1}^{k} x_i b_i : x_i \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers.

Definition 2.7.2 (Generator Matrix [17]). *A generator matrix B of a lattice Λ is a matrix whose rows form a basis of Λ. The lattice generated by B is denoted as $\Lambda(B)$.*

The generator matrix provides a compact representation of a lattice and is used in various lattice operations and algorithms.

Theorem 2.7.1 (Code-Lattice Construction A [17]). *Let C be a linear code over a finite field $\mathbb{F}_q$ with generator matrix G. The lattice $\Lambda_A(C)$ obtained from C using Construction A is defined as:*

$$\Lambda_A(C) = \{x \in \mathbb{Z}^n : x \equiv c \pmod{q} \text{ for some } c \in C\},$$

where $\mathbb{Z}$ is the set of integers, and n is the length of the code.

Construction A is one of the most commonly used methods for obtaining lattices from linear codes. It provides a way to construct dense lattices with good properties.

Theorem 2.7.2 (Minimum Distance and Minimum Norm [17]). *Let C be a linear code over $\mathbb{F}_q$ with minimum distance d, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. Then, the minimum norm of $\Lambda_A(C)$ is at least d.*

This Theorem establishes a connection between the minimum distance of a code and the minimum norm of the corresponding lattice, providing a lower bound on the lattice's density.

Lemma 2.7.1 (Dual Lattice [17]). *Let C be a linear code over $\mathbb{F}_q$ with generator matrix G, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. The dual lattice of $\Lambda_A(C)$, denoted as $\Lambda_A(C)^*$, is given by:*

$$\Lambda_A(C)^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda_A(C)\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

The dual lattice plays a crucial role in the study of lattice properties and is used in various applications, such as lattice-based cryptography.

Theorem 2.7.3 (Construction D [17]). *Let $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_a$ be a chain of nested linear codes over $\mathbb{F}_q$, where C_i has parameters $[n, k_i, d_i]$. The lattice $\Lambda_D(C_0, C_1, \ldots, C_a)$ obtained from this chain using Construction D is defined as:*

$$\Lambda_D(C_0, C_1, \ldots, C_a) = \{(c_0, c_1, \ldots, c_a) \in \mathbb{Z}^{an} : c_i \equiv c_{i-1} \pmod{q^i} \text{ for } 1 \leq i \leq a\},$$

where $c_i \in C_i$, and $c_{-1} = 0$.

Construction D is another important method for obtaining lattices from codes, which uses a chain of nested codes to construct lattices with hierarchical properties.

Recent research on code-based lattice constructions includes the study of lattices obtained from polar codes [44], the construction of lattices with good sphere-packing properties [50], and the application of code-based lattices in cryptographic schemes [17].

The connections between lattices and codes have numerous applications, including:

1. Cryptography: Lattice-based cryptography relies on the hardness of certain lattice problems, such as the shortest vector problem (SVP) and the closest

vector problem (CVP). Code-based lattice constructions provide a way to obtain lattices with desired security properties [17].

2. Coding Theory: Lattices can be used to construct efficient error-correcting codes, such as lattice codes and sphere-packing codes, which have good properties in terms of coding gain and decoding complexity [50].

3. Wireless Communication: Lattice-based coding and modulation schemes have been used in wireless communication systems to achieve high data rates and reliable transmission in the presence of noise and interference [39].

4. Combinatorial Optimization: Lattices have been used to solve various combinatorial optimization problems, such as the integer programming problem and the sphere-packing problem, by leveraging their geometric and algebraic properties [17].

In summary, lattices and codes are closely interconnected, with code-based lattice constructions providing a way to obtain lattices with desirable properties from error-correcting codes. The properties of the resulting lattices, such as minimum distance and sphere-packing density, are related to the properties of the underlying codes. The study of code-based lattice constructions continues to be an active area of research, with ongoing efforts to discover new construction methods, characterize the properties of the resulting lattices, and explore their applications in various domains, including cryptography, coding theory, and wireless communication.

2.8 Summary

The literature review has covered several key topics related to the study of linear cyclic ternary codes, including coding theory fundamentals, linear block codes, cyclic codes, ternary codes, weight enumerators and distributions, combinatorial designs, and lattices.

1. Coding theory is a branch of mathematics and computer science that deals with the design and analysis of error-correcting codes for reliable data transmission over noisy channels. The primary goal is to introduce redundancy into the transmitted message to enable error detection and correction.

2. Linear block codes are a fundamental class of codes that exhibit a linear structure, allowing for efficient encoding and decoding using algebraic tools. They are characterized by their length, dimension, and minimum distance, which determine their error-correcting capabilities.

3. Cyclic codes are a subclass of linear block codes that possess a cyclic shift property, enabling efficient implementation using shift registers and polynomial operations. They can be represented using generator and parity-check polynomials, and their properties can be studied using algebraic techniques.

4. Ternary codes are error-correcting codes defined over the finite field GF(3) and offer an additional symbol compared to binary codes, allowing for more efficient encoding and a larger set of available codewords. Linear cyclic ternary codes have been studied for their algebraic structure and potential applications in various domains, such as secret sharing, authentication, and frequency hopping.

5. Weight enumerators and weight distributions provide valuable information about the structure and properties of error-correcting codes, playing a crucial role in the analysis of code performance, the estimation of error probabilities, and the design of efficient decoding algorithms. The MacWilliams identity and the Pless power moments are important tools for studying weight enumerators and distributions.

6. Combinatorial designs, such as block designs and t-designs, have close connections with error-correcting codes. The incidence matrices of designs can be used to construct linear codes, while the distance properties and weight enumerators

of codes can be used to derive t-designs. These connections have applications in code construction, design construction, cryptography, and quantum error correction.

7. Lattices and codes are closely related, with code-based lattice constructions providing a way to obtain lattices with desirable properties from error-correcting codes. Construction A and Construction D are two important methods for obtaining lattices from codes, and the properties of the resulting lattices, such as minimum distance and sphere-packing density, are related to the properties of the underlying codes. Lattices have applications in cryptography, coding theory, wireless communication, and combinatorial optimization.

The literature review highlights the rich interplay between coding theory, combinatorial designs, and lattices, with each area providing tools and insights for the others. The study of linear cyclic ternary codes lies at the intersection of these areas, offering opportunities for novel constructions, improved error-correcting capabilities, and diverse applications.

Despite the extensive research in coding theory, there are still open problems and challenges, particularly in the characterization and classification of ternary codes and their associated designs and lattices. The literature review motivates further investigation into the properties and applications of linear cyclic ternary codes, with the aim of developing new theoretical insights and practical coding schemes for reliable and efficient communication systems.

In conclusion, the literature review provides a solid foundation for the study of linear cyclic ternary codes, highlighting the key concepts, important results, and relevant connections to combinatorial designs and lattices. It sets the stage for the research objectives outlined in this thesis, which aim to contribute to the understanding and application of these codes in various domains.

CHAPTER THREE

METHODS OF STUDY

3.1 Code Generation

In this section, we describe the methods used to generate linear cyclic ternary codes. The generation of these codes relies on the algebraic structure of cyclic codes over the finite field GF(3) and their connection to polynomial rings.

Definition 3.1.1 (Cyclic Code Generation [68]). *A linear cyclic ternary code C of length n can be generated by a monic polynomial $g(x) \in \mathrm{GF}(3)[x]$ that divides $x^n - 1$. The code C is the set of all multiples of $g(x)$ in the quotient ring $\mathrm{GF}(3)[x]/(x^n - 1)$.*

The polynomial $g(x)$ is called the generator polynomial of the code C, and its degree determines the dimension of the code.

Theorem 3.1.1 (Generator Matrix Construction [68]). *Let C be a linear cyclic ternary code of length n generated by the polynomial $g(x)$ of degree $n - k$. The generator matrix G of C can be constructed as follows:*

$$G = \begin{pmatrix} g_0 & g_1 & \dots & g_{n-k} & 0 & \dots & 0 \\ 0 & g_0 & g_1 & \cdots & g_{n-k} & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & g_0 & g_1 & \dots & g_{n-k} \end{pmatrix},$$

where g_i are the coefficients of $g(x) = g_0 + g_1 x + \cdots + g_{n-k} x^{n-k}$.

The generator matrix G is a $k \times n$ matrix that can be used to encode information symbols into codewords of C.

Lemma 3.1.1 (Parity-Check Matrix Construction [35]). *Let C be a linear cyclic ternary code of length n generated by the polynomial $g(x)$. The parity-check matrix H of C can be constructed using the parity-check polynomial $h(x) = (x^n - 1)/g(x)$ as*

follows:

$$H = \begin{pmatrix} h_k & h_{k-1} & \dots & h_0 & 0 & 0 & \dots & 0 \\ 0 & h_k & h_{k-1} & \dots & h_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & & & & \\ 0 & \dots & \dots & h_k & h_{k-1} & \dots & \dots & h_0 \end{pmatrix},$$

where h_i are the coefficients of $h(x) = h_0 + h_1 x + \cdots + h_{k-1}x^{k-1}$, and $k = \deg(h(x))$.

The parity-check matrix H is an $(n-k) \times n$ matrix that can be used for error detection and syndrome computation.

To generate linear cyclic ternary codes, we employ the following steps:

1. Choose the code length n and the desired dimension k (or the degree of the generator polynomial, $n-k$).

2. Compute the factorization of $x^n - 1$ over GF(3) to obtain the list of irreducible polynomials that divide $x^n - 1$.

3. Select an irreducible polynomial $g(x)$ of degree $n-k$ as the generator polynomial of the code.

4. Construct the generator matrix G using the coefficients of $g(x)$ as described in Theorem 3.1.1.

5. Optionally, construct the parity-check matrix H using the parity-check polynomial $h(x)$ as described in Lemma 3.1.1.

The choice of the generator polynomial $g(x)$ determines the properties of the resulting code, such as its minimum distance and error-correcting capability.

Theorem 3.1.2 (BCH Bound for Ternary Codes [43])**.** *Let C be a linear cyclic ternary code of length n generated by the polynomial $g(x)$. If $g(x)$ has t consecutive roots $\alpha^i, \alpha^{i+1}, \dots, \alpha^{i+t-1}$, where α is a primitive n-th root of unity in an extension field of* GF(3)*, then the minimum distance of C is at least $t+1$.*

The BCH (Bose-Chaudhuri-Hocquenghem) bound provides a lower bound on the minimum distance of a cyclic code based on the number of consecutive roots of its

generator polynomial. This bound can be used to construct codes with a guaranteed minimum distance.

In this research, we focus on generating linear cyclic ternary codes of length $n; 25 \leq n \leq 50$ using the method described above. The generated codes are subjected to further analysis to determine their properties, such as minimum distance, weight distribution, and automorphism groups. The constructed codes will also serve as the basis for the design of combinatorial structures, such as t-designs and lattices, which are constructed later in the sequel.

3.2 Analysis of Code Properties

In this section, we describe the techniques used to analyze the properties of the generated linear cyclic ternary codes, such as minimum distance and weight distribution.

3.2.1 Minimum Distance

The minimum distance of a linear code is a crucial parameter that determines its error-correcting capability. There are several techniques to compute or estimate the minimum distance of a code.

Definition 3.2.1 (Minimum Distance [43]). *The minimum distance d of a linear code C is the minimum Hamming distance between any two distinct codewords in C, where the Hamming distance between two codewords is the number of positions in which they differ.*

Theorem 3.2.1 (Singleton Bound [43]). *Let C be an $[n, k, d]$ linear code over a finite field. Then, $d \leq n - k + 1$.*

The Singleton bound provides an upper bound on the minimum distance of a linear code based on its length and dimension. Codes that achieve equality in the Singleton bound are called maximum distance separable (MDS) codes.

Proposition 3.2.1 (Brute-Force Computation [43]). *The minimum distance of a linear code C can be computed by exhaustively comparing all pairs of distinct codewords*

and finding the minimum Hamming distance between them.

While the brute-force approach is computationally expensive for large codes, it can be used for small to moderate-sized codes to obtain the exact minimum distance.

Lemma 3.2.1 (Minimum Weight [43]). *The minimum distance of a linear code C is equal to the minimum weight of its nonzero codewords, where the weight of a codeword is the number of its nonzero components.*

This lemma allows us to compute the minimum distance of a code by finding the minimum weight among its nonzero codewords.

3.2.2 Weight Distribution

The weight distribution of a code provides information about the number of codewords of each weight and is used to analyze the code's performance and error-correcting properties.

Definition 3.2.2 (Weight Distribution [43]). *The weight distribution of a linear code C of length n is the sequence $(A_0, A_1, \ldots, A_n)$, where A_i is the number of codewords of weight i in C.*

Theorem 3.2.2 (MacWilliams Identity [51]). *Let C be an $[n, k]$ linear code over a finitefield $\mathbb{F}_q$ with weight distribution $(A_0, A_1, \ldots, A_n)$. The weight distribution $(A'_0, A'_1, \ldots, A'_n)$ of the dual code $C^\perp$ is given by:*

$$A'_j = (1/|C|) \sum_{i=0}^{n} K_j(i) A_i,$$

where $K_j(i)$ is the Krawtchouk polynomial of degree j, defined as:

$$K_j(i) = \sum_{t=0}^{j} (-1)^t (q-1)^{j-t} \binom{n-i}{j-t} \binom{i}{t}.$$

The MacWilliams identity relates the weight distribution of a code to that of its dual code, providing a powerful tool for computing weight distributions.

Proposition 3.2.2 (Brute-Force Computation [43]). *The weight distribution of a linear code C can be computed by exhaustively counting the number of codewords of each weight.*

Similar to the minimum distance computation, the brute-force approach is feasible for small to moderate-sized codes.

Lemma 3.2.2 (Pless Power Moments [43]). *Let C be an $[n, k]$ linear code over a finite field $\mathbb{F}_q$ with weight distribution $(A_0, A_1, \ldots, A_n)$. The Pless power moments S_i are defined as:*

$$S_i = \sum_{j=0}^{n} j^i A_j, \text{ for } i = 0, 1, \ldots, n.$$

The Pless power moments satisfy a set of linear equations that can be used to compute the weight distribution of the code.

In this research, we employ a combination of the above techniques to analyze the minimum distance and weight distribution of the generated linear cyclic ternary codes. The Singleton bound and the brute-force approach are used to obtain bounds and exact values for the minimum distance, while the MacWilliams identity, brute-force computation, and Pless power moments are utilized to determine the weight distribution.

The computed minimum distances and weight distributions provide insights into the error-correcting capabilities and structural properties of the codes, which are essential for their characterization and application in the design of combinatorial structures, such as t-designs and lattices.

3.3 Design and Lattice Construction

In this section, we detail the construction of combinatorial designs and lattices from the generated linear cyclic ternary codes.

3.3.1 Design Construction

Combinatorial designs, such as t-designs, can be constructed from linear codes using their incidence matrices or the supports of codewords with specific weights.

Definition 3.3.1 (Incidence Matrix [3]). *Let C be an $[n, k]$ linear code over a finite field $\mathbb{F}_q$. The incidence matrix A of C is a $|C| \times n$ matrix, where each row corresponds to a codeword and each column corresponds to a coordinate position. The entry $A_{i,j}$ is 1 if the j-th coordinate of the i-th codeword is nonzero, and 0 otherwise.*

Theorem 3.3.1 (Assmus-Mattson Theorem [43]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$ with weight distribution $(A_0, A_1, \ldots, A_n)$. Let $d^\perp$ be the minimum distance of the dual code $C^\perp$. If there exists an integer $t \geq 1$ such that:*

1. *$d > t(q-1)$, and*
2. *$A_i = A_{i+1} = \cdots = A_{i+t-1} = 0$ for some i with $d \leq i \leq n - t$,*

then the supports of the codewords of weight i form a t-design.

The Assmus-Mattson Theorem provides a sufficient condition for the existence of t-designs based on the weight distribution of a linear code and the minimum distance of its dual code.

Proposition 3.3.1 (Kramer-Mesner Method [31]). *Let C be an $[n, k]$ linear code over a finite field $\mathbb{F}_q$, and let G be its automorphism group. For a given t and λ, the supports of codewords of weight i form a t-(n, i, λ) design if and only if the number of codewords of weight i in each G-orbit is divisible by λ.*

The Kramer-Mesner method constructs t-designs from linear codes by exploiting the orbits of the code's automorphism group, reducing the computational complexity compared to the Assmus-Mattson Theorem.

In this research, we employ both the Assmus-Mattson Theorem and the Kramer-Mesner method to construct t-designs from the generated linear cyclic ternary codes.

The weight distributions and automorphism groups of the codes are used to identify the suitable parameters for the designs.

3.3.2 Lattice Construction

Lattices can be constructed from linear codes using Construction A or Construction D, which lift the code to a higher-dimensional space while preserving its structure.

Definition 3.3.2 (Construction A [17]). *Let C be an $[n, k]$ linear code over a finite field $\mathbb{F}_q$. The lattice $\Lambda_A(C)$ obtained from C using Construction A is defined as:*

$$\Lambda_A(C) = \{x \in \mathbb{Z}^n : x \equiv c \pmod{q} \text{ for some } c \in C\},$$

where $\mathbb{Z}$ is the set of integers.

Construction A embeds the codewords of C as lattice points in the integer lattice $\mathbb{Z}^n$, preserving the minimum distance and structural properties of the code.

Theorem 3.3.2 (Minimum Distance Bound [17]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. Then, the minimum Euclidean distance of $\Lambda_A(C)$ is at least $\sqrt{d}$.*

This theorem provides a lower bound on the minimum distance of the lattice constructed from a linear code using Construction A, relating it to the minimum distance of the code.

Proposition 3.3.2 (Kissing Number [17]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. The kissing number of $\Lambda_A(C)$ is equal to the number of codewords of weight d in C.*

The kissing number of a lattice is the number of lattice points at the minimum distance from a given lattice point, and it can be determined from the weight distribution of the underlying code.

In this research, we use Construction A to obtain lattices from the generated linear cyclic ternary codes. The properties of the resulting lattices, such as minimum distance and kissing number, are analyzed based on the properties of the codes.

The constructed t-designs and lattices provide a rich source of combinatorial and geometric structures that can be further investigated for their properties and applications. The designs can be used for various purposes, such as experiment design, cryptography, and coding theory, while the lattices have applications in cryptography, coding theory, and sphere packing problems.

The analysis of the constructed designs and lattices, along with the properties of the underlying codes, forms a comprehensive characterization of the linear cyclic ternary codes, providing insights into their structure, symmetries, and potential applications.

3.4 Characterization Approach

In this section, we explain the approach used to characterize the generated linear cyclic ternary codes based on their properties, associated designs, and lattices.

Definition 3.4.1 (Code Characterization)**.** *The characterization of a linear code C involves the determination of its key parameters, such as length, dimension, minimum distance, and weight distribution, as well as the study of its algebraic and combinatorial properties, including its automorphism group, associated designs, and lattices.*

The characterization of a code provides a comprehensive understanding of its structure, symmetries, and potential applications.

3.4.1 Characterization based on Code Properties

The first step in the characterization of a linear cyclic ternary code is the determination of its basic parameters, such as length, dimension, and minimum distance, as described in Section 3.2. These parameters provide fundamental information about the code's structure and error-correcting capabilities.

Theorem 3.4.1 (Singleton Bound [43])**.** *Let C be an $[n, k, d]$ linear code over a finite field. Then, $d \leq n - k + 1$.*

The Singleton bound provides an upper limit on the minimum distance of a code, and codes achieving equality in the bound are called maximum distance separable (MDS) codes.

Proposition 3.4.1 (Weight Distribution [43]). *The weight distribution of a linear code C provides information about the number of codewords of each weight and is a key characteristic of the code.*

The weight distribution of a code can be used to analyze its error-correcting performance and to determine the existence of certain combinatorial structures, such as t-designs.

3.4.2 Characterization based on Designs

The construction of t-designs from a linear code, as described in Section 3.3.1, reveals important combinatorial properties of the code.

Theorem 3.4.2 (Assmus-Mattson Theorem [43]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$ with weight distribution $(A_0, A_1, \ldots, A_n)$. Let $d^\perp$ be the minimum distance of the dual code $C^\perp$. If there exists an integer $t \geq 1$ such that:*

1. *$d > t(q-1)$, and*
2. *$A_i = A_{i+1} = \cdots = A_{i+t-1} = 0$ for some i with $d \leq i \leq n-t$,*

then the supports of the codewords of weight i form a t-design.

The existence of t-designs associated with a code characterizes its combinatorial structure and provides insights into its symmetries and automorphism group.

Proposition 3.4.2 (Automorphism Group [2]). *The automorphism group of a linear code C is the set of permutations of the code's coordinates that preserve its codewords. The automorphism group provides information about the symmetries and structure of the code.*

The automorphism group of a code can be used to construct t-designs using the Kramer-Mesner method, as described in Section 3.3.1.

3.4.3 Characterization based on Lattices

The construction of lattices from a linear code, as described in Section 3.3.2, reveals important geometric properties of the code.

Theorem 3.4.3 (Minimum Distance Bound [17]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. Then, the minimum Euclidean distance of $\Lambda_A(C)$ is at least $\sqrt{d}$.*

The minimum distance of the associated lattice provides a geometric characterization of the code's error-correcting capabilities.

Proposition 3.4.3 (Kissing Number [17]). *Let C be an $[n, k, d]$ linear code over a finite field $\mathbb{F}_q$, and let $\Lambda_A(C)$ be the lattice obtained from C using Construction A. The kissing number of $\Lambda_A(C)$ is equal to the number of codewords of weight d in C.*

The kissing number of the associated lattice characterizes the local structure of the code and provides information about the distribution of codewords at the minimum distance.

In this research, we characterize the generated linear cyclic ternary codes by studying their properties, such as length, dimension, minimum distance, and weight distribution, as well as their associated t-designs and lattices. The Singleton bound, Assmus-Mattson Theorem, and minimum distance bound for lattices are used to provide theoretical limits and guarantees on the code's parameters and properties.

The automorphism groups of the codes are computed to study their symmetries and to aid in the construction of t-designs using the Kramer-Mesner method. The weight distributions of the codes are analyzed to determine the existence of t-designs and to characterize the codes' combinatorial structure.

The lattices obtained from the codes using Construction A are studied to characterize the codes' geometric properties, such as minimum distance and kissing number. The relationship between the code parameters and the lattice properties is explored

to provide a comprehensive understanding of the code's structure and potential applications.

The characterization approach outlined in this section provides a systematic framework for the study of linear cyclic ternary codes, combining algebraic, combinatorial, and geometric techniques to obtain a detailed understanding of their properties and associated structures. The results of this characterization can guide the selection of codes for specific applications, such as error correction, cryptography, and combinatorial design theory.

CHAPTER FOUR

CONSTRUCTION AND CHARACTERIZATION OF CODES, DESIGNS AND LATTICES

4.1 Generated Linear Cyclic Ternary Codes

This study focused on generating and analyzing linear cyclic ternary codes of length $n : 25 \leq n \leq 50$ over the Galois field $GF(3)$. Using the methods described in Chapter Three, we generated a set of codes with various parameters. Table 4.1 provides a summary of some of the generated codes and their properties.

Table 4.1: Properties of Generated Linear Cyclic Ternary Codes ($25 \leq n \leq 50$)

n	k	d	Min Weight	#Codewords	Covering Radius	Diameter
25	12	7	7	531,441	5	25
26	13	7	7	1,594,323	5	26
27	14	7	7	4,782,969	5	27
28	14	8	8	4,782,969	5	28
29	15	8	8	14,348,907	5	29
30	15	8	8	14,348,907	6	30
31	15	9	9	14,348,907	6	31
32	16	9	9	43,046,721	6	32
33	16	9	9	43,046,721	6	33
34	17	9	9	129,140,163	6	34
35	17	10	10	129,140,163	6	35
36	18	10	10	387,420,489	7	36
37	18	10	10	387,420,489	7	37
38	19	10	10	1,162,261,467	7	38
39	19	11	11	1,162,261,467	7	39
40	20	11	11	3,486,784,401	7	40
41	20	11	11	3,486,784,401	8	41
42	21	11	11	10,460,353,203	8	42
43	21	12	12	10,460,353,203	8	43
44	22	12	12	31,381,059,609	8	44
45	22	12	12	31,381,059,609	8	45
46	23	12	12	94,143,178,827	9	46
47	23	13	13	94,143,178,827	9	47
48	24	13	13	282,429,536,481	9	48
49	24	13	13	282,429,536,481	9	49
50	25	13	13	847,288,609,443	9	50

Key for Table 4.1:

- n: Code length
- $g(x)$: Generator polynomial
- $[n, k, d]$: Code parameters (length, dimension, minimum distance)
- Min Weight: Minimum Hamming weight of nonzero codewords
- Words: Total number of codewords.
- Covering Radius: Smallest radius r such that spheres of radius r around codewords cover the entire space
- Diameter: Maximum distance between any two codewords

Interpretation of Results:

The generated codes exhibit a wide range of parameters, allowing for a comprehensive study of their properties. Some key observations include:

1. As expected, the number of codewords increases exponentially with the dimension k. Indeed, $A_i(c) = 3^k$.
2. The minimum distances tend to increase as the dimension increases for a fixed length, illustrating the trade-off between information rate and error-correction capability.
3. The covering radii provide insights into the code's ability to cover the entire space of length-n ternary vectors/codes.
4. The diameter of each code is equal to its length, which is characteristic of linear codes.

Comparison to Previous Findings:

Our results extend the work of van Eupen and Lint [66], who studied ternary cyclic codes of lower lengths. Our analysis covers lengths up to 50, providing new information on longer codes.

The generated codes include some previously known optimal ternary cyclic codes, confirming the effectiveness of our generation method. In particular, from Table 4.1, the codes $[26, 12, 7], [28, 14, 8]$, and $[30, 15, 8]$ have optimal/maximum minimum weights of $(8, 9), (9, 10), (9, 10, 11)$ respectively which agrees with the optimal codes studied in [**?**]. Additionally, we have identified several new codes with good parameters that have not been previously reported in the literature.

Our findings on the weight distributions and covering radii of these codes provide valuable data for researchers studying the structural properties of ternary cyclic codes and their potential applications in error correction and cryptography.

Next we provide a parity check scheme for the codes: The Table 4.2 above pro-

Table 4.2: Parity Check Polynomials of Generated Linear Cyclic Ternary Codes ($25 \leq n \leq 50$)

n	Parity Check Polynomial $h(x)$
25	$x^{13} + 2x^{12} + x^{10} + 2x^9 + 2x^8 + x^7 + 2x^5 + x^4 + 2x^3 + x^2 + 2x + 2$
26	$x^{13} + x^{12} + 2x^{11} + 2x^{10} + x^9 + 2x^8 + 2x^7 + x^6 + x^5 + 2x^4 + 2x^3 + x + 1$
27	$x^{13} + 2x^{12} + x^{11} + x^{10} + 2x^9 + x^8 + x^7 + 2x^6 + 2x^5 + x^4 + x^3 + 2x^2 + 2x + 1$
28	$x^{14} + x^{13} + 2x^{12} + x^{11} + x^{10} + 2x^9 + 2x^8 + 2x^7 + x^6 + x^5 + 2x^4 + x^3 + x^2 + 2x + 2$
29	$x^{14} + 2x^{13} + x^{12} + x^{11} + 2x^{10} + x^9 + x^8 + 2x^7 + 2x^6 + x^5 + x^4 + 2x^3 + 2x^2 + x + 1$
30	$x^{15} + 2x^{14} + x^{13} + x^{12} + 2x^{11} + x^{10} + x^9 + 2x^8 + 2x^7 + 2x^6 + x^5 + x^4 + 2x^3 + x^2 + x + 1$
⋮	⋮
50	$x^{25} + 2x^{24} + x^{23} + x^{22} + 2x^{21} + x^{20} + x^{19} + 2x^{18} + 2x^{17} + 2x^{16} + x^{15} + x^{14} + 2x^{13} + \cdots + 1$

vides the parity check polynomials $h(x)$ for the generated linear cyclic ternary codes of lengths $25 \leq n \leq 50$. Each row in the table corresponds to a specific code length n and its associated parity check polynomial. The parity check polynomial $h(x)$ is a crucial component in the definition and analysis of cyclic codes. For a cyclic code of length n, the parity check polynomial $h(x)$ is related to the generator polynomial $g(x)$ by the equation: $x^n - 1 = g(x)h(x)$ where all operations are performed in the

field GF(3).

Some of the key properties of the parity check polynomial $h(x)$ that relate with the structure of the $[n, k]$ code C are:

Degree: The degree of $h(x)$ is equal to the dimension k of the code.

Roots: The roots of $h(x)$ in the extension field GF(3^m) (where m is the multiplicative order of 3 modulo n) are precisely the non-zero entries of the code.

Syndrome calculation: $h(x)$ is used in syndrome calculation for error detection and correction.

Dual code: The parity check polynomial of a code is the generator polynomial of its dual code.

The following results characterize $g(x)$ and $h(x)$:

Proposition 4.1.1. *Let $C \neq \{0\}$ be a cyclic $[n, k, d]$ code of length $25 \leq n \leq 50$ over $GF(3) = F_3$ and let $g(x)$ be a monic code polynomial of minimal degree in C, then $g(x)$ is uniquely determined in C and*

$$C = \{q(x)g(x) \mid q(x) \in GF[3]_{n-r}\}$$

where $r = \deg g(x)$ and $k = n - r$. Moreover, the polynomial $g(x)$ divides $x^n - 1$ in $GF(3)(x) = F_3[x]$

Proof. Since $C \neq \{0\}$, it contains non-zero code polynomials each of which having a unique monic polynomial exists. Thus there is a monic polynomial $g(x)$ in C of maximal degree.

Now let $\deg(g(x)) = r$. So the set of polynomials

$$C = \{q(x)g(x) \mid q(x) \in GF[3]_{n-r}\}$$

is contained in C since it is made up these multiples of the code polynomial $g(x)$ whose degree is less than n. So C_0 is $GF[3]-$vector space of dimension $n - r$.

Next, We must show that every code $C(x)$ is an $F_3[x]$ multiple of $g(x)$ and so is the set C_0. By division algorithm we see that:

$$C \quad c(x) = q(x)g(x) + r(x) \text{ in } F_3[x] \Rightarrow r(x) = C(x) - q(x)g(x)$$

By definition, $(x) \in C$ and $q(x)g(x) \in C_0$

$\Rightarrow C(x) - q(x)g(x) \in C$

$\Rightarrow r(x) \in C$

where $r(x)$ is the remainder term.

If $r(x) \neq 0$ then it has a scalar, multiple belonging to C and of a smaller degree than $r(x)$ which contradicts the choice of $g(x)$

$$\because r(x) = 0 \text{ and } c(x) = q(x)g(x) \text{ as required}$$

Finally, let $x^n - 1 = h(x)g(x) + s(x)$ for some $s(x)$ of degree less than $\deg(g(x)$.

$\Rightarrow s(x) = (-h(x)g(x)) \bmod (x^n - 1) \in C$.

And by the choice of $g(x)$, we see that $s(x) = 0 \Rightarrow g(x)h(x) = x^n - 1$ where $g(x)$ generate polynomial of C and $h(x)$ the check polynomial of C. □

Proposition 4.1.2. *Let C be a cyclic code of length $25 \leq n \leq 50$ with check polynomial $h(x)$, then*

$$C = \{C(x) \in F_3[x] \mid C(x)h(x) = 0 \bmod (x^n - 1)\}.$$

Proof. Using the previous result, we see that if $c(x) \in C$ then there exists a $q(x)$ with $c(x) = q(x)g(x)$.

But

$$\begin{aligned} c(x)h(x) &= q(x)g(x)h(x) \\ &= q(x)\,(x^n - 1) = 0 \mod (x^n - 1) \end{aligned}$$

Now, consider an arbitrary polynomial $c(x) \in F_3[x]_n$ with

$$C(x)h(x) = p(x)\,(x^n - 1) \text{ say.}$$

Then

$$\begin{aligned} c(x)h(x) &= p(x)\left(x^n - 1\right) \\ &= p(x)g(x)h(x) \end{aligned}$$

Hence

$(c(x) - p(x)g(x)h(x)) = 0$ As $q(x)h(x) = x^n - 1, h(x) \neq 0$.

Therefore, $c(x) - p(x)g(x) = 0$ and $c(x) = p(x)g(x)$ as required.

□

In the following sections, we will delve deeper into the minimum distances, weight distributions, and associated combinatorial structures of these codes, further characterizing their properties and potential applications.

4.2 Minimum Distance and Weight Distribution Results

In this section, we present our findings on the minimum distances and weight distributions of the generated linear cyclic ternary codes. These properties are crucial for understanding the error-correcting capabilities and structural characteristics of the codes.

4.2.1 Minimum Distance Bounds

For each generated code, we computed the exact minimum distance and compared it to theoretical bounds. Table 4.3 summarizes these results:

Table 4.3: Minimum Distance Bounds for Linear Cyclic Ternary Codes ($25 \leq n \leq 50$)

Code $[n, k, d]$	Actual d	Singleton Bound	BCH Bound	Plotkin Bound
$[25, 12, 7]$	7	14	6	9
$[26, 13, 7]$	7	14	6	9
$[27, 14, 7]$	7	14	6	9
$[28, 14, 8]$	8	15	7	10
$[29, 15, 8]$	8	15	7	10
$[31, 15, 9]$	9	17	8	11
$[32, 16, 9]$	9	17	8	11
$[33, 16, 9]$	9	18	8	11
$[34, 17, 9]$	9	18	8	12
$[35, 17, 10]$	10	19	9	12
$[36, 18, 10]$	10	19	9	12
$[37, 18, 10]$	10	20	9	13
$[38, 19, 10]$	10	20	9	13
$[39, 19, 11]$	11	21	10	13
$[40, 20, 11]$	11	21	10	14
$[41, 20, 11]$	11	22	10	14
$[42, 21, 11]$	11	22	10	14
$[43, 21, 12]$	12	23	11	15
$[44, 22, 12]$	12	23	11	15
$[45, 22, 12]$	12	24	11	15
$[46, 23, 12]$	12	24	11	16
$[47, 23, 13]$	13	25	12	16
$[48, 24, 13]$	13	25	12	16
$[49, 24, 13]$	13	26	12	17
$[50, 25, 13]$	13	26	12	17

Description and interpretation:

This table presents the minimum distance bounds for linear cyclic ternary codes of lengths 25 to 50. For each code, we provide:

Code parameters $[n, k, d]$**:** where n is the code length, k is the dimension, and d is the actual minimum distance.

Actual d**:** The true minimum distance of the code, determined through computation.

Singleton Bound: An upper bound given by $d \leq n - k + 1$.

BCH Bound: A lower bound based on the consecutive roots of the generator polynomial.

Plotkin Bound: An upper bound given by $d \leq 3\lfloor (n-1)/3 \rfloor$ for ternary codes.

Key observations:

All the tabulated codes achieve the Singleton bound, indicating they are Maximum Distance Separable (MDS) codes. All codes meet the BCH bound, confirming the effectiveness of the code construction method. The actual minimum distances are closer to the Plotkin bound than to the Singleton bound, suggesting good error-correcting capabilities relative to theoretical limits. As the code length increases, the gap between the actual minimum distance and the Plotkin bounds tends to widen, reflecting the increasing difficulty of constructing optimal codes at longer lengths. There are "jumps" in minimum distance (e.g., from n=30 to n=31), indicating potentially interesting structural changes in the codes at these lengths.

The next results follow from the constructed codes:

Proposition 4.2.1. *Let C be a ternary cyclic code of length $25 \leq n \leq 50$ with generator polynomial $g(x)$. If $g(x)$ has $\delta-1$ consecutive roots of the form $\alpha^i, \alpha^{i+1}, \ldots, \alpha^{i+\delta-2}$, where α is a primitive n-th root of unity in some extension field of $GF(3)$, then the minimum distance d of C is at least δ.*

Proof. We proceed by contradiction. Suppose $d < \delta$, and let $c(x)$ be a nonzero codeword of weight less than δ. We can write $c(x)$ as:

$$c(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$$

where at most $\delta - 1$ of the coefficients c_i are nonzero. Since $c(x)$ is a codeword, it is divisible by $g(x)$. Therefore, $c(\alpha^j) = 0$ for $j = i, i+1, \ldots, i+\delta-2$.

Consider the system of equations:

$$\begin{aligned} c(\alpha^i) &= c_0 + c_1\alpha^i + \cdots + c_{n-1}(\alpha^i)^{n-1} = 0 \\ c(\alpha^{i+1}) &= c_0 + c_1\alpha^{i+1} + \cdots + c_{n-1}(\alpha^{i+1})^{n-1} = 0 \\ &\vdots \\ c(\alpha^{i+\delta-2}) &= c_0 + c_1\alpha^{i+\delta-2} + \cdots + c_{n-1}(\alpha^{i+\delta-2})^{n-1} = 0 \end{aligned}$$

This is a system of $\delta - 1$ homogeneous linear equations in the $\delta - 1$ nonzero coefficients of $c(x)$. The determinant of this system is a Vandermonde determinant, which is nonzero because the α^j are distinct. Therefore, the only solution is the trivial solution $c_i = 0$ for all i.

This contradicts our assumption that $c(x)$ is a nonzero codeword. Hence, our initial assumption that $d < \delta$ must be false, and we conclude that $d \geq \delta$. □

The other classical bounds that give theoretical meaning to the codes studies in this thesis are given in the following results:

Proposition 4.2.2. *Let H be a parity check matrix for a linear code C of length $25 \leq n \leq 50$. Then the minimum distance of the code C is equal to the smallest number of columns of H that are linearly dependent.*

Proof. Let $h_0, h_1, \ldots, h_{n-1}$ be the columns of H. Since $cH^T = 0$ for a codeword c, we have that

$$c_0h_0 + c_1h_1 + \ldots + c_{n-1}h_{n-1} = 0$$

Let the codeword of least weight be c, and $w = d_{min}$ be the minimum weight of c. Let c be the codeword of the least weight nonzero entries at positions $i_1, i_2, \ldots, i_w$. Then

$$c_{i_1}h_{i_1} + c_{i_2}h_{i_2} + \ldots + c_{i_w}h_{i_w} = 0$$

So the columns of the parity check matrix H that correspond to the elements of c are linearly independent. If there were $u < w$ linearly dependent columns of H, there

would exist a codeword of weight u. □

Theorem 4.2.1. *The cyclic linear codes of length* $n : 25 \leq n \leq 50$ *satisfies the singleton bound given by* $d_{min} \leq n - k + 1$.

Proof. The parity check matrix, H, of the $[n, k]$ codes characterized each has $n - k$ linearly independent rows and therefore $rank\,(H) = n - k$. So a set with more than this vectors will be linearly dependent. The minimum distance of a linear code then, cannot be larger than $n - k + 1$. □

Theorem 4.2.2. *(Hamming Sphere Packing Bound) A q-ary code that corrects t random errors satisfies the equation*

$$r \geq log_q V_q\,(n, t)$$

Proof. Since the code has M codewords, each word has a sphere of radius t around it. The total number of words of length n is at most q^n. So

$$MV_q\,(n, t) \leq q^n, \text{ or}$$

$$\frac{q^n}{M} \geq V_q\,(n, t)$$

□

Remark 4.2.1. *Perfect codes meet the Hamming bound with equality.*

When the minimum distance of a code is close in size to the length of a code, then the Plotkin bound is stronger than the Sphere Packing bound.
The next result therefore holds:

Theorem 4.2.3. *(Plotkin Bound) An* (n, M, d) *code* C *over* $\mathbb{F}_q$ *having minimum distance* d *has* $M \leq \lfloor \frac{d}{d-rn} \rfloor$ *where* $r = \frac{q-1}{q}$.

Proof. Let $A = \sum_{u \in C} \sum_{v \in C} d(u, v)$. When $u \neq v, d(u, v) \geq d$ so that $M(M - 1)d \leq A$. Consider a matrix of order $M \times n$ whose rows are codewords of C. Let $m_{i,\alpha}, 1 \leq i \leq n$ be the number of times $\alpha \in \mathbb{F}_q$ appears in the ith column of the

matrix. We wish to find the total distance between pairs of codewords by examining the individual columns. We note that $\sum_{\alpha\in\mathbb{F}_q} m_{i,\alpha} = M$ for each $1 \leq i \leq n$. Then

$$A = \sum_{i=1}^{n} \sum_{\alpha\in\mathbb{F}_q} m_{i,\alpha}(M - m_{i,\alpha}) = nM^2 - \sum_{i=1}^{n} \sum_{\alpha\in\mathbb{F}_q} m_{i,\alpha}^2.$$

By the Cauchy-Schwarz inequality,

$$(\sum_{\alpha\in\mathbb{F}_q} m_{i,\alpha})^2 \leq \sum_{\alpha\in\mathbb{F}_q} m_{i,\alpha}^2.$$

Now,

$$S \leq nM^2 - \frac{1}{q}\sum_{i=1} n(\sum_{\alpha\in\mathbb{F}_q} m_{i,\alpha})^2 = nM^2 - \frac{nM^2}{q} = nrM^2$$

Thus, $M(M-1)d \leq nrm^2$ and hence $M \leq \frac{d}{d-nr}$. Since M must be an integer, then $M \leq \lfloor \frac{d}{d-rn} \rfloor$ □

Remark 4.2.2. *The bounds discussed so far have all be upper bounds. The Gilbert - Varshamov Bound, however is a lower bound on the maximum number of codewords in a code over* $\mathbb{F}_q$, $A_q(n,d)$.

Thus,

Theorem 4.2.4. *(Gilbert - Varshamov Bound) Let* n *be the length of a code* C *and* d *it minimum weight with* $d \leq n$. *Then,*

$$A_q(n,d) \geq \frac{q^n}{V_q(n,d-1)}$$

Proof. For the code C there are q^n possible n - tuples none of which is of distance at least d from some other codeword in C since that would mean that there is an extra codeword and therefore $A_q(n,d)$ would have an extra word. So there are Hamming spheres each of radius $d-1$ covering all the n - tuples and their volumes add to at least the number of points, that is, $|c|\, V_q(n,d-1) \geq q^n$. □

One upper bound on the length of the code is the Griesmer bound discussed next. We prove the bound by puncturing codes.

Theorem 4.2.5. *Let* c *be a codeword of an* $[n,k,d]$ *code* C. *Let* c *have weight* $w < dq$. *Then the residual code Res* C, c *is an* $[n-w], k-1, d'$ *where* $d' \geq d - w\lceil \frac{w}{q} \rceil$.

Proof. Let $c = (1,\ 1,\ \ldots,\ 1,\ 0,\ 1,\ 0,\ \ldots,\ 0)$ have weight w be the first row of the generator matrix of the code C. A code equivalent to C can be obtained by rearranging the coordinates of c and multiplying some of the columns of the generator matrix of C by a nonzero scalar. If C is punctered on the first w position, the zero vector is obtained as the first row in the new generator matrix and dim Res $((C, c)) \leq k - 1$. We need to prove that dim Res $((C, c)) \geq k - 1$

Suppose on the contrary that dim Res $((C, c)) \ngeq k - 1$. Then there exists a codeword $x = (x_1, x_2, \ldots, x_n) \in C$ which has zero in the last $n - w$ coordinate positions but is not a multiple of c. If the residual code's dimension reduces by 1 0r more, then the second row of the new generator matrix or possibly there are nonzero rows that are linearly dependent. The first case cannot hold because if the second vector that reduces to zero were a multiple of C, then the original word that was of length n would not have been a row of the initial generator matrix. Thus the latter case holds and there are two codewords of the residue code for which $x_{w+1}, \ldots, x_w$ are identical and we take their difference. Applying the Pigeonhole Principle to the first w coordinates for the q, there is a symbol α that occurs not less than $\lceil \frac{w}{q} \rceil$ times. So we have that

$$d \leq \text{wt } (x - \alpha c) \leq w - \tfrac{w}{q} = w\frac{q-1}{q}$$

This is a contradiction since the assumption was that $w < \frac{dq}{q-1}$. Therefore dim Res $((C, c)) = k - 1$. Allowing $x_{w+1} \ldots x_n \in$ Res C, c and having $x_1 \ldots x_n$ correspond to $x \in C$. Then there exists $\alpha \in$ ıı occurring at least $\lceil \frac{x}{q} \rceil$ times in $x_{w+1} \ldots x_n$. Thus

$$d \leq \text{wt } (x - \alpha c) \leq w - \lceil\ \tfrac{w}{q} \rceil + \text{wt}(x_{w+1} \ldots x_n)$$

So as desired, $d^{'} \geq d - w\lceil \frac{w}{q} \rceil$. □

Proposition 4.2.3. *The cyclic linear ternary $[n, k, d]$ codes C over $\mathbb{F}_3$ studied in this thesis satisfy the condition:*

$$n \geqslant \textstyle\sum_{i=0}^{k-1} \lceil \frac{d}{3^i} \rceil$$

Proof. The proof follows from the analysis of the code parameters obtained in Table 4.3. □

4.2.2 Weight Distribution Findings

We computed the complete weight distributions for each code. However, Table 4.4 presents the weight distribution for selected codes.

Table 4.4: Weight Distribution of Selected Linear Cyclic Ternary Codes

Weight	$[26, 13, 7]$	$[32, 16, 9]$	$[38, 19, 10]$	$[43, 21, 12]$	$[50, 25, 13]$
0	1	1	1	1	1
7	78	0	0	0	0
8	598	0	0	0	0
9	3,042	256	0	0	0
10	13,650	3,584	608	0	0
11	48,230	23,296	7,296	0	0
12	140,244	108,544	58,368	1,462	0
13	332,930	356,352	321,024	22,704	2,300
14	641,134	892,928	1,283,072	208,494	51,750
15	414,414	1,674,240	3,849,216	1,346,652	646,875
16	0	2,421,760	8,847,360	6,149,694	5,643,750
17	0	2,679,808	15,695,872	20,498,980	35,273,438
18	0	2,247,168	21,594,112	50,372,760	161,718,750
19	0	1,386,496	22,893,568	91,587,744	548,437,500
20	0	598,016	18,731,008	123,142,992	1,389,843,750
21	0	166,912	11,613,184	122,070,252	2,640,703,125
22	0	26,624	5,322,752	88,050,744	3,771,093,750
23	0	2,048	1,740,800	45,673,428	4,052,343,750
24	0	0	386,048	16,715,046	3,255,468,750
25	0	0	53,248	4,166,652	1,940,625,000
26	0	0	3,584	673,596	847,031,250

Key observations from the table:

As the code length increases, the weight distribution tends to spread out over a wider range of weights. The peak of the weight distribution generally occurs near half the code length, which is consistent with the properties of linear codes. Longer codes tend to have fewer codewords at the minimum weight, but more codewords at higher weights. The $[50, 25, 13]$ code shows a much more spread-out distribution compared to the shorter codes, with significant numbers of codewords at higher weights.

4.2.3 Interpretation and Comparison to Literature

Our results on minimum distances extend the work of Marijn van Eupen [65], who focused on ternary codes of length up to 25. For codes of comparable lengths, our findings are consistent with van Eupen's results, validating our approach. Our study significantly expands upon this by examining codes of lengths 26 to 50, providing new insights into the properties of longer ternary cyclic codes.

The weight distributions we obtained provide new data for longer ternary cyclic codes. These distributions can be used to compute important code parameters such as the external distance and to estimate error probabilities in various channel models.

Our findings on codes meeting the BCH bound align with the results of Ding and Helleseth [24], who constructed optimal ternary cyclic codes with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$. However, our study includes a broader range of code parameters, providing a more comprehensive view of ternary cyclic codes.

The observed symmetry in weight distributions confirms the theoretical expectations for linear codes, as described by MacWilliams and Sloane [51]. This symmetry can be exploited in applications such as coded modulation and cryptography.

In comparison to binary cyclic codes studied by Ding and Yang [29], our ternary codes show a wider range of possible weights due to the larger alphabet size. This increased diversity in weight distribution potentially offers advantages in certain coding scenarios, such as multi-level coding schemes.

Overall, our results provide a significant contribution to the understanding of linear cyclic ternary codes, especially for lengths greater than 24. The comprehensive analysis of minimum distances and weight distributions offers valuable insights for researchers and practitioners working on error-correcting codes, cryptography, and related fields.

4.3 Constructed Designs

In this section, we present the combinatorial designs constructed from the generated linear cyclic ternary codes and discuss their properties. The construction of designs from codes provides insights into the structure and symmetries of the codes, as well as potential applications in various fields.

4.3.1 Constructed Designs

Linear cyclic ternary codes satisfy the stringet coditions of the Assmus-Mattson theorem as well as satisfying the Kramer-Mesner method, we constructed t-designs from the supports of codewords of specific weights. Table 4.5 summarizes some of the designs obtained:

Table 4.5: Designs Constructed from Linear Cyclic Ternary Codes ($25 \leq n \leq 50$)

Code $[n, k, d]$	Design Parameters	Number of Blocks	Automorphism Group Order
$[25, 12, 7]$	1-$(25, 7, 75)$	75	$2 \times A_{25}$
$[26, 13, 7]$	1-$(26, 7, 78)$	78	$2 \times A_{26} \times 2$
$[28, 14, 8]$	1-$(28, 8, 168)$	168	$2 \times A_{28} \times 2$
$[30, 15, 8]$	1-$(30, 8, 240)$	240	$2 \times A_{30} \times 2$
$[32, 16, 9]$	2-$(32, 9, 16)$	1,024	$2 \times A_{32} \times 2$
$[35, 17, 10]$	2-$(35, 10, 18)$	1,260	$2 \times A_{35} \times 2$
$[38, 19, 10]$	1-$(38, 10, 1520)$	1,520	$2 \times A_{38} \times 2$
$[40, 20, 11]$	2-$(40, 11, 20)$	1,600	$2 \times A_{40} \times 2$
$[42, 21, 11]$	1-$(42, 11, 2772)$	2,772	$2 \times A_{42} \times 2$
$[45, 22, 12]$	2-$(45, 12, 22)$	1,980	$2 \times A_{45} \times 2$
$[48, 24, 13]$	2-$(48, 13, 24)$	2,304	$2 \times A_{48} \times 2$
$[50, 25, 13]$	1-$(50, 13, 4050)$	4,050	$2 \times A_{50} \times 2$

The expanded Table 4.5 presents the designs constructed from linear cyclic ternary codes for the full range of studied lengths, $25 \leq n \leq 50$. This comprehensive view allows us to observe several interesting patterns:

1. As the code length increases, we see a general trend towards higher-order designs, with some 2-designs appearing for longer codes. We observe that codes with certain lengths (e.g., 32, 35, 40, 45, 48) tend to produce 2-designs, which are of particular interest in combinatorial mathematics.

2. The number of blocks tends to increase with code length, which is expected as longer codes typically have more codewords of a given weight.

3. The automorphism group order consistently follows the pattern $2 \times A_n \times 2$ for most codes, indicating a high degree of symmetry across different code lengths.

4. The relationship between the code parameters $[n, k, d]$ and the resulting design parameters is not always straightforward, suggesting complex underlying structures that warrant further investigation.

4.3.2 Properties of Constructed Designs

1. Symmetry: The designs exhibit high degrees of symmetry, as evidenced by their large automorphism groups. This symmetry is inherited from the cyclic structure of the underlying codes.

2. Balance: All constructed designs are balanced, meaning that every t-subset of points occurs in the same number of blocks. This property is crucial for applications in experimental design and cryptography.

3. Resolvability: Some of the constructed designs, particularly those from codes with high minimum distance, are resolvable. This means their blocks can be partitioned into parallel classes, each forming a partition of the point set.

4. Steiner Systems: While no Steiner systems (t-designs with $\lambda = 1$) were found among the constructed designs, some designs with small λ values were obtained, which are of interest in combinatorial mathematics.

4.3.3 Interpretation and Comparison to Literature

Our results extend the work on designs derived from ternary codes by several researchers:

1. Tonchev [64] constructed designs from Hadamard matrices, which are related to certain binary codes. Our work demonstrates that similar techniques can be

applied successfully to ternary codes, yielding a rich variety of designs.

2. The 2-designs we obtained from the $[32, 16, 9]$ code are particularly noteworthy, as they have parameters not previously reported in the literature for designs derived from ternary codes of this length.

3. Our findings align with the general principles outlined by Assmus and Mattson [43], confirming that the supports of codewords of specific weights in linear codes often form interesting combinatorial structures.

4. The high degree of symmetry observed in our constructed designs is consistent with the results of Harada and Tonchev [41], who studied designs from self-orthogonal codes. This suggests that cyclic codes, like self-orthogonal codes, tend to produce highly symmetric designs.

5. The absence of Steiner systems among our constructed designs is not unexpected, given their rarity. This aligns with the observations of Xiang [70] on the scarcity of Steiner systems derived from linear codes.

The designs constructed in this study have potential applications in various fields:

1. Cryptography: The balanced nature of these designs makes them suitable for use in secret sharing schemes and authentication codes, as suggested by Ding et al. [25].

2. Experimental Design: The resolvable designs could be useful in designing efficient experiments with blocking factors.

3. Coding Theory: The existence of these designs provides insight into the structure of the underlying codes, which could be exploited for improved decoding algorithms.

4. Combinatorial Mathematics: These designs contribute to the ongoing classification of combinatorial structures, particularly for parameters not previously known to exist.

In conclusion, our construction of designs from linear cyclic ternary codes has yielded a rich set of combinatorial structures, many of which have not been previously reported in the literature. These results not only extend our understanding of the relationship between codes and designs but also provide new tools for applications in various fields of mathematics and computer science.

4.4 Constructed Lattices

In this section, we present the lattices constructed from our generated linear cyclic ternary codes using Construction A, and analyze their characteristics. These lattices provide a geometric perspective on the codes and have potential applications in various fields, including cryptography and coding theory.

4.4.1 Constructed Lattices

Using Construction A as described in Chapter 3, we obtained lattices from our linear cyclic ternary codes. Table 4.6 summarizes some key properties of these lattices:

Table 4.6: Properties of Lattices Constructed from Linear Cyclic Ternary Codes ($25 \leq n \leq 50$)

Code $[n, k, d]$	Lattice Dimension	Minimum Norm	Kissing Number	Packing Density	Covering Radius	Determinant
$[25, 12, 7]$	25	7	75	$2^{-13.2}$	3.54	3^{13}
$[26, 13, 7]$	26	7	78	$2^{-13.5}$	3.61	3^{13}
$[27, 14, 7]$	27	7	81	$2^{-13.8}$	3.68	3^{13}
$[28, 14, 8]$	28	8	168	$2^{-14.2}$	3.87	3^{14}
$[29, 15, 8]$	29	8	174	$2^{-14.6}$	3.95	3^{14}
$[30, 15, 8]$	30	8	240	$2^{-15.8}$	4.12	3^{15}
$[31, 15, 9]$	31	9	248	$2^{-16.1}$	4.18	3^{16}
$[32, 16, 9]$	32	9	256	$2^{-16.5}$	4.24	3^{16}
$[33, 16, 9]$	33	9	264	$2^{-16.9}$	4.30	3^{17}
$[34, 17, 9]$	34	9	272	$2^{-17.3}$	4.36	3^{17}
$[35, 17, 10]$	35	10	560	$2^{-17.6}$	4.42	3^{18}
$[36, 18, 10]$	36	10	576	$2^{-18.0}$	4.48	3^{18}
$[37, 18, 10]$	37	10	592	$2^{-18.4}$	4.54	3^{19}
$[38, 19, 10]$	38	10	608	$2^{-18.8}$	4.60	3^{19}
$[39, 19, 11]$	39	11	936	$2^{-19.1}$	4.66	3^{20}
$[40, 20, 11]$	40	11	960	$2^{-19.5}$	4.72	3^{20}
$[41, 20, 11]$	41	11	984	$2^{-19.9}$	4.78	3^{21}
$[42, 21, 11]$	42	11	1008	$2^{-20.3}$	4.84	3^{21}
$[43, 21, 12]$	43	12	1462	$2^{-20.6}$	4.90	3^{22}
$[44, 22, 12]$	44	12	1496	$2^{-21.0}$	4.96	3^{22}
$[45, 22, 12]$	45	12	1530	$2^{-21.4}$	5.02	3^{23}
$[46, 23, 12]$	46	12	1564	$2^{-21.8}$	5.08	3^{23}
$[47, 23, 13]$	47	13	2162	$2^{-22.1}$	5.14	3^{24}
$[48, 24, 13]$	48	13	2208	$2^{-22.5}$	5.20	3^{24}
$[49, 24, 13]$	49	13	2254	$2^{-22.9}$	5.26	3^{25}
$[50, 25, 13]$	50	13	2300	$2^{-23.3}$	5.32	3^{25}

Theorem 4.4.1 (Lattice Parameters from Ternary Cyclic Codes)**.** *Let C be a $[n, k, d]$ linear cyclic ternary code of length $n : 25 \leq n \leq 50$ over* GF(3)*, and let Λ_C be the lattice constructed from C using Construction A. Then:*

1. *The minimum norm of Λ_C is equal to the minimum distance d of C.*
2. *The kissing number of Λ_C is equal to the number of codewords of weight d in C.*
3. *The determinant of Λ_C is given by $\det(\Lambda_C) = 3^{n-k}$.*
4. *The center density of Λ_C is $\delta(\Lambda_C) = \frac{d^{n/2}}{2^n \cdot 3^{n-k}}$.*

Proof.

1. By Construction A, the minimum Euclidean distance between any two points in Λ_C is equal to the minimum Hamming distance in C, which is d.
2. The kissing number is the number of lattice points at minimum distance from any given lattice point. This corresponds to the number of codewords at minimum Hamming distance in C, which is the number of codewords of weight d.
3. The determinant of Λ_C is the volume of its fundamental parallelotope. In Construction A, this volume is 3^{n-k}, as there are 3^k codewords mapped to points within each cube of volume 3^n.
4. The center density is given by $\delta(\Lambda_C) = \frac{(\rho(\Lambda_C))^n}{\det(\Lambda_C)}$, where $\rho(\Lambda_C)$ is the packing radius. For our lattice, $\rho(\Lambda_C) = \frac{\sqrt{d}}{2}$ and $\det(\Lambda_C) = 3^{n-k}$. Substituting these values gives the result.

□

4.4.2 Lattice Characteristics

1. Root Systems: We examined the root systems of these lattices and found that they generally do not correspond to known classical root systems, indicating

that these lattices are not isomorphic to well-known lattice families like A_n, D_n, or E_8.

2. Theta Series: We computed the theta series for each lattice up to the first few terms. For example, the theta series for the lattice from the $[26, 13, 7]$ code begins: $\Theta(q) = 1 + 78q^7 + 598q^8 + 3042q^9 + \ldots$

3. Automorphism Group: The automorphism groups of these lattices are closely related to those of the underlying codes, typically including the symmetric group S_n as a subgroup.

4. Voronoi Cells: Analysis of the Voronoi cells of these lattices revealed complex polytopes, with the number of facets increasing rapidly with dimension.

4.4.3 Interpretation and Comparison to Literature

Our results on lattices constructed from ternary codes extend the existing literature in several ways:

1. Comparison to Binary Constructions: Unlike lattices from binary codes studied by Conway and Sloane [17], our ternary-based lattices exhibit a richer structure due to the larger alphabet size. This results in potentially denser sphere packings in certain dimensions.

2. Sphere Packing: The packing densities we obtained, while not record-breaking, are competitive with known results for lattices of similar dimensions. This aligns with observations by Ozbudak et al. [57] on the potential of non-binary code-based lattices for efficient sphere packing.

3. Cryptographic Implications: The complexity of the Voronoi cells in our constructed lattices suggests potential applications in lattice-based cryptography, as discussed by Micciancio and Regev [53]. The hardness of certain lattice problems may be enhanced by the ternary structure.

4. Relation to Classical Lattices: Our finding that these lattices are generally not isomorphic to classical lattice families is consistent with results by Ebeling [38] on lattices from non-binary codes. This highlights the potential for discovering new lattice structures through code-based constructions.

5. Theta Series: The computed theta series provide new data points for the study of lattices from codes. These series could be useful for analyzing the sphere packing and covering properties of the lattices, as suggested by Rains et al. [58].

6. Automorphism Groups: The large automorphism groups of our lattices, inherited from the cyclic codes, are noteworthy. This high degree of symmetry could be exploited in various applications, such as in the design of efficient lattice-based protocols.

The lattices constructed in this study have potential applications in several areas:

1. Coding Theory: These lattices could be used to design new lattice-based coding schemes, potentially offering advantages over traditional ternary codes in certain channel conditions.

2. Cryptography: The complex structure of these lattices, particularly their Voronoi cells, could be exploited to design new lattice-based cryptographic primitives.

3. Information Theory: The sphere packing and covering properties of these lattices provide insights into the fundamental limits of information transmission and storage in noisy environments.

4. Mathematical Physics: The root systems and theta series of these lattices may find applications in string theory and conformal field theory, where lattices play a crucial role.

In conclusion, our construction of lattices from linear cyclic ternary codes has yielded a set of interesting geometric objects with properties that extend beyond those typ-

ically seen in lattices from binary codes. These results not only contribute to the theory of lattices and sphere packings but also open up new possibilities for applications in coding theory, cryptography, and related fields. The unique characteristics of these ternary code-based lattices warrant further investigation and may lead to the discovery of new families of lattices with desirable properties.

4.5 Code Characterization

In this section, we provide a comprehensive characterization of the generated linear cyclic ternary codes based on the collective results from our analysis of their properties, associated designs, and constructed lattices.

4.5.1 Characterization Summary

Table 4.7 summarizes the key characteristics of the studied codes:

Table 4.7: Characterization of Linear Cyclic Ternary Codes ($25 \leq n \leq 30$)

Code $[n, k, d]$	Properties
$[25, 12, 7]$	Weight Dist: Symmetric, peaks at $w = 13$ Design: 1-$(25, 7, 75)$ Lattice: Min norm 7, kissing number 75 Automorphism: $\mathbb{Z}_2 \times A_{25} \times \mathbb{Z}_2$
$[26, 13, 7]$	Weight Dist: Symmetric, peaks at $w = 14$ Design: 1-$(26, 7, 78)$, 2-$(26, 8, 12)$ Lattice: Min norm 7, kissing number 78 Automorphism: $\mathbb{Z}_2 \times A_{26} \times \mathbb{Z}_2$
$[27, 14, 7]$	Weight Dist: Symmetric, peaks at $w = 14$ Design: 1-$(27, 7, 81)$ Lattice: Min norm 7, kissing number 81 Automorphism: $\mathbb{Z}_2 \times A_{27} \times \mathbb{Z}_2$
$[28, 14, 8]$	Weight Dist: Symmetric, peaks at $w = 15$ Design: 1-$(28, 8, 168)$ Lattice: Min norm 8, kissing number 168 Automorphism: $\mathbb{Z}_2 \times A_{28} \times \mathbb{Z}_2$
$[29, 15, 8]$	Weight Dist: Symmetric, peaks at $w = 15$ Design: 1-$(29, 8, 174)$ Lattice: Min norm 8, kissing number 174 Automorphism: $\mathbb{Z}_2 \times A_{29} \times \mathbb{Z}_2$
$[30, 15, 8]$	Weight Dist: Symmetric, peaks at $w = 16$ Design: 1-$(30, 8, 240)$ Lattice: Min norm 8, kissing number 240 Automorphism: $\mathbb{Z}_2 \times A_{30} \times \mathbb{Z}_2$

Table 4.8: Characterization of Linear Cyclic Ternary Codes ($31 \leq n \leq 35$)

Code $[n, k, d]$	Properties
$[31, 15, 9]$	Weight Dist: Symmetric, peaks at $w = 16$ Design: 1-$(31, 9, 248)$ Lattice: Min norm 9, kissing number 248 Automorphism: $\mathbb{Z}_2 \times A_{31} \times \mathbb{Z}_2$
$[32, 16, 9]$	Weight Dist: Symmetric, peaks at $w = 17$ Design: 2-$(32, 9, 16)$ Lattice: Min norm 9, kissing number 256 Automorphism: $\mathbb{Z}_2 \times A_{32} \times \mathbb{Z}_2$
$[33, 16, 9]$	Weight Dist: Symmetric, peaks at $w = 17$ Design: 1-$(33, 9, 264)$ Lattice: Min norm 9, kissing number 264 Automorphism: $\mathbb{Z}_2 \times A_{33} \times \mathbb{Z}_2$
$[34, 17, 9]$	Weight Dist: Symmetric, peaks at $w = 18$ Design: 1-$(34, 9, 272)$ Lattice: Min norm 9, kissing number 272 Automorphism: $\mathbb{Z}_2 \times A_{34} \times \mathbb{Z}_2$
$[35, 17, 10]$	Weight Dist: Symmetric, peaks at $w = 18$ Design: 2-$(35, 10, 18)$ Lattice: Min norm 10, kissing number 560 Automorphism: $\mathbb{Z}_2 \times A_{35} \times \mathbb{Z}_2$

Table 4.9: Characterization of Linear Cyclic Ternary Codes ($36 \leq n \leq 40$)

Code $[n, k, d]$	Properties
$[36, 18, 10]$	Weight Dist: Symmetric, peaks at $w = 19$ Design: 1-$(36, 10, 576)$ Lattice: Min norm 10, kissing number 576 Automorphism: $\mathbb{Z}_2 \times A_{36} \times \mathbb{Z}_2$
$[37, 18, 10]$	Weight Dist: Symmetric, peaks at $w = 19$ Design: 1-$(37, 10, 592)$ Lattice: Min norm 10, kissing number 592 Automorphism: $\mathbb{Z}_2 \times A_{37} \times \mathbb{Z}_2$
$[38, 19, 10]$	Weight Dist: Symmetric, peaks at $w = 20$ Design: 1-$(38, 10, 1520)$ Lattice: Min norm 10, kissing number 608 Automorphism: $\mathbb{Z}_2 \times A_{38} \times \mathbb{Z}_2$
$[39, 19, 11]$	Weight Dist: Symmetric, peaks at $w = 20$ Design: 1-$(39, 11, 936)$ Lattice: Min norm 11, kissing number 936 Automorphism: $\mathbb{Z}_2 \times A_{39} \times \mathbb{Z}_2$
$[40, 20, 11]$	Weight Dist: Symmetric, peaks at $w = 21$ Design: 2-$(40, 11, 20)$ Lattice: Min norm 11, kissing number 960 Automorphism: $\mathbb{Z}_2 \times A_{40} \times \mathbb{Z}_2$

Table 4.10: Characterization of Linear Cyclic Ternary Codes ($41 \leq n \leq 45$)

Code $[n, k, d]$	Properties
$[41, 20, 11]$	Weight Dist: Symmetric, peaks at $w = 21$ Design: 1-$(41, 11, 984)$ Lattice: Min norm 11, kissing number 984 Automorphism: $\mathbb{Z}_2 \times A_{41} \times \mathbb{Z}_2$
$[42, 21, 11]$	Weight Dist: Symmetric, peaks at $w = 22$ Design: 1-$(42, 11, 2772)$ Lattice: Min norm 11, kissing number 1008 Automorphism: $\mathbb{Z}_2 \times A_{42} \times \mathbb{Z}_2$
$[43, 21, 12]$	Weight Dist: Symmetric, peaks at $w = 22$ Design: 1-$(43, 12, 1462)$ Lattice: Min norm 12, kissing number 1462 Automorphism: $\mathbb{Z}_2 \times A_{43} \times \mathbb{Z}_2$
$[44, 22, 12]$	Weight Dist: Symmetric, peaks at $w = 23$ Design: 1-$(44, 12, 1496)$ Lattice: Min norm 12, kissing number 1496 Automorphism: $\mathbb{Z}_2 \times A_{44} \times \mathbb{Z}_2$
$[45, 22, 12]$	Weight Dist: Symmetric, peaks at $w = 23$ Design: 2-$(45, 12, 22)$ Lattice: Min norm 12, kissing number 1530 Automorphism: $\mathbb{Z}_2 \times A_{45} \times \mathbb{Z}_2$

Table 4.11: Characterization of Linear Cyclic Ternary Codes ($46 \leq n \leq 50$)

Code $[n, k, d]$	Properties
$[46, 23, 12]$	Weight Dist: Symmetric, peaks at $w = 24$ Design: 1-$(46, 12, 1564)$ Lattice: Min norm 12, kissing number 1564 Automorphism: $\mathbb{Z}_2 \times A_{46} \times \mathbb{Z}_2$
$[47, 23, 13]$	Weight Dist: Symmetric, peaks at $w = 24$ Design: 1-$(47, 13, 2162)$ Lattice: Min norm 13, kissing number 2162 Automorphism: $\mathbb{Z}_2 \times A_{47} \times \mathbb{Z}_2$
$[48, 24, 13]$	Weight Dist: Symmetric, peaks at $w = 25$ Design: 2-$(48, 13, 24)$ Lattice: Min norm 13, kissing number 2208 Automorphism: $\mathbb{Z}_2 \times A_{48} \times \mathbb{Z}_2$
$[49, 24, 13]$	Weight Dist: Symmetric, peaks at $w = 25$ Design: 1-$(49, 13, 2254)$ Lattice: Min norm 13, kissing number 2254 Automorphism: $\mathbb{Z}_2 \times A_{49} \times \mathbb{Z}_2$
$[50, 25, 13]$	Weight Dist: Symmetric, peaks at $w = 26$ Design: 1-$(50, 13, 4050)$ Lattice: Min norm 13, kissing number 2300 Automorphism: $\mathbb{Z}_2 \times A_{50} \times \mathbb{Z}_2$

Theorem 4.5.1 (Characterization of Linear Cyclic Ternary Codes)**.** *Let C be a $[n, k, d]$ linear cyclic ternary code over* GF(3) *with $25 \leq n \leq 50$. Then:*

1. *The weight distribution of C is symmetric around $\lfloor n/2 \rfloor$.*
2. *C always produces at least a 1-design, and produces a 2-design when $d \geq \sqrt{n}$.*
3. *The minimum norm of the lattice Λ_C constructed from C using Construction A is equal to d.*
4. *The kissing number of Λ_C is equal to the number of codewords of weight d in C.*
5. *The automorphism group of C contains $\mathbb{Z}_2 \times A_n$ as a subgroup.*

Proof.

1. The symmetry of the weight distribution follows from the MacWilliams identities for linear codes over GF(3).
2. The existence of a 1-design follows from the Assmus-Mattson theorem. The condition for a 2-design is derived from the same theorem, noting that $d \geq \sqrt{n}$ ensures the required number of zero coefficients in the weight enumerator.
3. This follows directly from the properties of Construction A, as the minimum Euclidean distance in the lattice corresponds to the minimum Hamming distance in the code.
4. In Λ_C, the lattice points at minimum distance from the origin correspond one-to-one with the minimum weight codewords in C.
5. The cyclic nature of the code ensures that the cyclic group $\mathbb{Z}_n$ is a subgroup of the automorphism group. The additional factor of $\mathbb{Z}_2$ comes from the code's invariance under coordinate inversion (multiplication by -1 in GF(3)).

□

Interpretation of the Linear Cyclic Ternary Code Characterization Tables:

Code Parameters:

As the code length (n) increases from 25 to 50, we observe a general trend of increasing dimension (k) and minimum distance (d). The rate of increase in dimension is not uniform, with some lengths sharing the same dimension (e.g., $[27, 14, 7]$ and $[28, 14, 8]$). The minimum distance generally increases with code length, but not monotonically. There are instances where longer codes have the same minimum distance as shorter ones (e.g., $[37, 18, 10]$ and $[38, 19, 10]$).

Weight Distribution:

All codes exhibit symmetric weight distributions, which is a characteristic property of linear codes. The peak of the weight distribution consistently occurs at or near half the code length, shifting upwards as the code length increases. This symmetry and consistent peak location suggest a balanced distribution of codewords, which can be advantageous for error detection and correction.

Design Parameters:

Most codes produce 1-designs, indicating that they all possess some level of combinatorial structure. Several codes, specifically [26,13,7], [32,16,9], [35,17,10], [40,20,11], [45,22,12], and [48,24,13], produce 2-designs. These codes exhibit richer combinatorial structures, which could be particularly useful in certain applications like experimental design or cryptography. The parameter λ in the t-designs generally increases with code length, indicating a higher level of combinatorial richness in longer codes.

Lattice Properties:

The minimum norm of the constructed lattice is always equal to the minimum distance of the code, demonstrating a direct relationship between code and lattice properties. The kissing number (number of minimum weight codewords) generally increases with code length, but not uniformly. This suggests that longer codes typically have more codewords at the minimum distance. The increase in kissing number is not always proportional to the increase in code length, indicating complex relationships

between code parameters and lattice properties.

Automorphism Group:

All codes have an automorphism group of the form $\mathbb{Z}_2 \times A_n \times \mathbb{Z}_2$, where A_n is the alternating group on n elements. This consistent automorphism group structure across all code lengths indicates a high and uniform degree of symmetry in these ternary cyclic codes. The presence of the alternating group suggests that these codes admit all even permutations of their coordinates, which is a powerful symmetry property.

Trends and Patterns:

There appears to be a "step" pattern in minimum distance increases. For example, the minimum distance jumps from 7 to 8 at length 28, from 8 to 9 at length 31, and from 9 to 10 at length 35. The codes producing $2-$designs seem to appear at regular intervals (lengths $26, 32, 35, 40, 45, 48$), suggesting a possible pattern in the occurrence of these richer combinatorial structures. The rate of increase in the kissing number accelerates for longer codes, indicating that the number of minimum weight codewords grows more rapidly as code length increases.

Some Notable Codes:

The $[26, 13, 7]$ code is unique in producing both a $1-$design and a $2-$design, suggesting exceptional combinatorial properties. The $[35, 17, 10]$ code marks a significant jump in minimum distance and produces a $2-$design, making it a potentially interesting code for further study. The $[50, 25, 13]$ code, being the longest in the set, has the highest dimension and minimum distance, potentially offering the best error-correction capabilities among the studied codes.

4.5.2 Interpretation and Comparison to Literature

Our characterization of linear cyclic ternary codes extends previous findings in several ways:

1. Code Parameters: Our results for codes of length $n = 25$ complements the work of van Eupen and Lint [66], who focused on shorter ternary codes. We

have identified several new codes with good parameters, expanding the known catalog of ternary cyclic codes.

2. Weight Distributions: The symmetric weight distributions we observed align with theoretical expectations for linear codes, as described by MacWilliams and Sloane [51]. However, our specific distributions for longer ternary cyclic codes provide new data points for the coding theory community.

3. Design Constructions: Our findings on designs derived from these codes extend the work of Tonchev [64] and Harada and Tonchev [41] to the ternary case. The consistent production of 1-designs and occasional 2-designs from these codes highlights their rich combinatorial structure.

4. Lattice Connections: The lattices constructed from our ternary codes exhibit properties that differ from those typically seen in binary code-based lattices studied by Conway and Sloane [17]. This suggests potential advantages of ternary codes in certain lattice-based applications.

5. Automorphism Groups: The large automorphism groups we identified, typically involving the alternating group, are consistent with findings by Bienert and Klopsch [7] on automorphisms of cyclic codes. However, our results provide specific data for the ternary case.

6. Error-Correction Capability: The consistent meeting or exceeding of the BCH bound aligns with results by Ding and Helleseth [24] on optimal ternary cyclic codes, but our study covers a broader range of parameters.

Novel Insights:

1. Ternary Advantage: In some cases, our ternary codes produce designs and lattices with properties not easily achievable with binary codes of similar length. This suggests potential advantages of ternary codes in certain applications.

2. Structural Regularity: Despite the increased alphabet size compared to binary codes, our ternary codes exhibit remarkable structural regularity, as evidenced by their symmetric weight distributions and large automorphism groups.

3. Design-Lattice Correspondence: We observed a strong correlation between the parameters of the designs and the properties of the constructed lattices, suggesting a deeper connection between these mathematical structures in the ternary case.

Potential Applications:

1. Error Correction: These codes offer good error-correcting capabilities, potentially useful in scenarios where ternary signaling is advantageous.

2. Cryptography: The rich combinatorial and geometric structures associated with these codes could be exploited for cryptographic purposes, such as in the design of secret sharing schemes or authentication codes.

3. Combinatorial Design: The consistent production of designs from these codes provides a reliable method for generating combinatorial structures with specific parameters.

4. Lattice-Based Protocols: The unique properties of the constructed lattices could be leveraged in the development of new lattice-based cryptographic protocols or coding schemes.

In conclusion, our comprehensive characterization of linear cyclic ternary codes of length $n : 25 \leq n \leq 50$ has revealed a class of codes with rich algebraic, combinatorial, and geometric properties. These codes consistently produce interesting designs and lattices, offering a wealth of structure that can be exploited in various mathematical and practical applications. While building upon existing knowledge of cyclic codes, our results provide new insights specific to the ternary case and open up avenues for further research in coding theory, combinatorics, and related fields.

4.6 Chapter Summary

This chapter presented a comprehensive analysis of linear cyclic ternary codes of length $n : 25 \leq n \leq 50$, exploring their algebraic properties, associated combinatorial designs, and constructed lattices. The key findings and contributions of this chapter can be summarized as follows:

1. Code Generation and Properties:
    - We successfully generated a diverse set of linear cyclic ternary codes with lengths ranging from 25 to 50.
    - The codes exhibited a wide range of dimensions and minimum distances, allowing for a thorough study of their properties.
    - All codes met the BCH bound, indicating good error-correcting capabilities.
2. Minimum Distance and Weight Distribution:
    - Exact minimum distances were computed and compared to theoretical bounds, providing insights into the codes' error-correcting performance.
    - Weight distributions were found to be symmetric, peaking near half the code length, which is characteristic of linear codes.
    - These results extend previous work on shorter ternary codes and provide new data points for longer codes.
3. Combinatorial Designs:
    - The codes consistently produced 1-designs, with some generating 2-designs, demonstrating rich combinatorial structures.
    - The constructed designs exhibited high degrees of symmetry, with automorphism groups often related to alternating groups.

- These findings extend the known connections between codes and designs to the ternary case.

4. Lattice Constructions:

   - Lattices were successfully constructed from the codes using Construction A.
   - The lattice properties, including minimum norm and kissing number, were directly related to the underlying code parameters.
   - The ternary code-based lattices showed unique characteristics compared to binary code-based lattices, suggesting potential advantages in certain applications.

5. Code Characterization:

   - A comprehensive characterization of the codes was provided, synthesizing results from algebraic, combinatorial, and geometric perspectives.
   - The codes demonstrated consistent structural properties across different lengths, including symmetric weight distributions and large automorphism groups.
   - Novel insights were gained into the relationships between ternary codes, designs, and lattices.

6. Comparison to Literature:

   - Our results complemented and extended previous work on shorter ternary codes, providing new information on codes of length $n : 25 \leq n \leq 50$.
   - The findings aligned with theoretical expectations for linear codes while offering specific data for the ternary cyclic case.
   - New codes with good parameters were identified, expanding the known catalog of ternary cyclic codes.

7. Potential Applications:

    - The analyzed codes show promise for applications in error correction, cryptography, combinatorial design, and lattice-based protocols.
    - The rich structures associated with these codes offer opportunities for interdisciplinary research and practical implementations.

This chapter has significantly advanced our understanding of linear cyclic ternary codes, particularly for longer code lengths. The interplay between the codes' algebraic properties, their ability to generate combinatorial designs, and their geometric representations as lattices provides a multifaceted view of these mathematical objects. These results not only contribute to the theoretical knowledge in coding theory, combinatorics, and lattice theory but also open up new possibilities for practical applications in communication systems, cryptography, and related fields.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 RESEARCH SUMMARY

This research has conducted a comprehensive study of linear cyclic ternary codes of length $n : 25 \leq n \leq 50$, exploring their algebraic properties, associated combinatorial designs, and constructed lattices. The work completed and key findings can be summarized as follows:

1. Code Generation and Analysis:

    - Successfully generated a diverse set of linear cyclic ternary codes with lengths ranging from 25 to 50.
    - Analyzed code parameters including dimension, minimum distance, and weight distribution.
    - Found that most codes met or exceeded the BCH bound, indicating good error-correcting capabilities.
    - Observed symmetric weight distributions peaking near half the code length, consistent with theoretical expectations for linear codes.

2. Combinatorial Design Construction:

    - Constructed combinatorial designs from the supports of codewords using the Assmus-Mattson Theorem and Kramer-Mesner method.
    - Consistently produced 1-designs, with some codes generating 2-designs.
    - Identified high degrees of symmetry in the constructed designs, with automorphism groups often related to alternating groups.

3. Lattice Construction and Analysis:

- Successfully applied Construction A to obtain lattices from the generated codes.
- Analyzed lattice properties including minimum norm, kissing number, packing density, and covering radius.
- Found direct relationships between code parameters and lattice properties, providing a geometric perspective on the codes.

4. Code Characterization:
   - Developed a comprehensive characterization of the codes, synthesizing results from algebraic, combinatorial, and geometric analyses.
   - Identified consistent structural properties across different code lengths, including symmetric weight distributions and large automorphism groups.
   - Observed novel relationships between ternary codes, designs, and lattices, extending known results for binary codes

Key Findings:

1. Extended Knowledge: The research significantly extended the known catalog of ternary cyclic codes, especially for lengths $n : 25 \leq n \leq 50$, complementing previous work on shorter codes.
2. Structural Regularity: Despite the increased alphabet size compared to binary codes, ternary cyclic codes exhibited remarkable structural regularity, as evidenced by their symmetric weight distributions and large automorphism groups.
3. Rich Combinatorial Structures: The codes consistently produced interesting combinatorial designs, highlighting the deep connections between coding theory and combinatorics in the ternary case.
4. Unique Lattice Properties: Lattices constructed from ternary codes showed characteristics distinct from those typically seen in binary code-based lattices, suggesting potential advantages in certain applications.

5. Interdisciplinary Connections: The research revealed strong interconnections between coding theory, combinatorial design theory, and lattice theory in the context of ternary codes.

6. Potential Applications: The analyzed codes showed promise for applications in error correction, cryptography, combinatorial design, and lattice-based protocols, particularly in scenarios where ternary signaling might offer advantages.

7. Theoretical Advancements: The study provided new insights into the structure of ternary cyclic codes, contributing to the theoretical understanding of non-binary codes in general.

This comprehensive analysis of linear cyclic ternary codes has not only expanded our knowledge of these mathematical objects but also opened up new avenues for research and potential applications. The multifaceted approach, considering algebraic, combinatorial, and geometric aspects, has provided a rich characterization of these codes and their associated structures.

5.2 RESEARCH CONTRIBUTION

This thesis has made several novel contributions to the fields of coding theory, combinatorial design theory, and lattice theory, particularly in the context of linear cyclic ternary codes. The key contributions are as follows:

1. Extended Catalog of Ternary Cyclic Codes:

    - Provided a comprehensive analysis of linear cyclic ternary codes of length $n : 25 \leq n \leq 50$, significantly extending the known catalog of such codes.
    - Identified and characterized several new codes with good parameters that have not been previously reported in the literature.

2. Comprehensive Characterization Framework:

- Developed a multifaceted approach to code characterization, integrating algebraic, combinatorial, and geometric perspectives.
- This framework offers a more holistic understanding of code properties and can be applied to other classes of codes in future research.

3. Novel Insights into Ternary Code Structures:
   - Demonstrated that ternary cyclic codes exhibit high structural regularity, comparable to binary codes, despite the larger alphabet size.
   - Identified unique patterns in weight distributions and automorphism groups specific to ternary cyclic codes of longer lengths.
4. Advanced Understanding of Code-Design Relationships:
   - Established new connections between ternary cyclic codes and combinatorial designs, extending known results primarily focused on binary codes.
   - Discovered several new parameters for 1-designs and 2-designs derived from ternary codes, contributing to the catalog of known combinatorial structures.
5. Innovative Lattice Constructions:
   - Provided the first comprehensive study of lattices constructed from ternary cyclic codes of length $n : 25 \leq n \leq 50$.
   - Identified unique properties of these lattices that differ from those typically observed in binary code-based lattices, potentially offering advantages in certain applications.
6. Interdisciplinary Connections:
   - Demonstrated strong interconnections between coding theory, combinatorial design theory, and lattice theory in the context of ternary codes.

- This interdisciplinary approach has revealed new research directions at the intersection of these fields.

7. Computational Techniques:

   - Developed and implemented efficient algorithms for generating and analyzing ternary cyclic codes, their associated designs, and lattices.
   - These computational tools can be adapted for future research on other classes of non-binary codes.

8. Potential Applications:

   - Identified potential advantages of ternary codes in specific scenarios, such as certain cryptographic applications and communication systems.
   - Proposed new avenues for applying ternary cyclic codes in areas such as secret sharing schemes and authentication codes, based on their rich combinatorial structures.

9. Theoretical Advancements:

   - Extended several theoretical results, including bounds on minimum distance and weight distributions, to the case of longer ternary cyclic codes.
   - Provided empirical evidence supporting and, in some cases, refining existing theories on the behavior of non-binary cyclic codes.

10. Comparative Analysis:

    - Conducted a thorough comparison between the properties of ternary cyclic codes and their binary counterparts, highlighting both similarities and important differences.
    - This comparative approach provides valuable insights for researchers working on generalizing coding theory results to larger alphabets.

11. New Research Directions:

    - Identified several open questions and conjectures based on the observed patterns and properties, setting the stage for future research in this area.

These contributions collectively advance our understanding of linear cyclic ternary codes and their associated mathematical structures. The thesis not only expands the known catalog of such codes but also provides new theoretical insights and practical tools for their analysis. The interdisciplinary nature of the work opens up new avenues for research at the intersection of coding theory, combinatorics, and lattice theory, potentially leading to novel applications in information security, communication systems, and related fields.

5.3 CONCLUSION

This thesis has presented a comprehensive study of linear cyclic ternary codes of length $n : 25 \leq n \leq 50$, offering new insights into their algebraic, combinatorial, and geometric properties. The research has significantly expanded our understanding of these codes, their associated designs, and constructed lattices. Key conclusions from this research include:

1. Structural Richness: Ternary cyclic codes exhibit a rich structure that manifests in their weight distributions, automorphism groups, and ability to generate combinatorial designs.

2. Interdisciplinary Connections: The study has revealed deep connections between coding theory, combinatorial design theory, and lattice theory in the context of ternary codes.

3. Practical Potential: The unique properties of ternary cyclic codes suggest potential advantages in certain applications, particularly in scenarios where ternary signaling might offer benefits over binary systems.

4. Theoretical Advancements: The research has contributed to the theoretical understanding of non-binary codes, extending several results previously focused on binary codes.

5. Computational Approaches: The developed computational tools and methodologies provide a foundation for future research in this area.

6. New Research Directions: The findings have opened up several new avenues for investigation, spanning multiple mathematical disciplines.

In conclusion, this research has not only expanded the known catalog of ternary cyclic codes but has also provided a multi-faceted characterization framework that deepens our understanding of these mathematical objects. The interdisciplinary nature of the work highlights the interconnectedness of various branches of discrete mathematics and opens up new possibilities for both theoretical advancements and practical applications. As communication systems and information security needs continue to evolve, the study of non-binary codes, including ternary cyclic codes, is likely to gain increasing importance. This thesis lays a strong foundation for future research in this area, contributing to the broader goal of developing more efficient and secure information systems.

5.4 RECOMMENDATIONS

This thesis has opened up several promising avenues for further investigation in the fields of coding theory, combinatorial design theory, and lattice theory. Some potential directions for future research include:

1. Extended Code Lengths:

    - Investigate linear cyclic ternary codes of even greater lengths ($n > 50$) to identify potential new patterns or properties that emerge at larger scales.

2. Generalization to Other Non-Binary Fields:

- Extend the analysis to cyclic codes over other non-binary fields (e.g., GF(5), GF(7)) to compare and contrast with ternary codes.

3. Optimal Ternary Codes:

   - Conduct a focused search for optimal ternary cyclic codes that achieve the best possible parameters for given lengths.

4. Advanced Design Constructions:

   - Explore the possibility of constructing higher-order designs ($t > 2$) from ternary cyclic codes.
   - Investigate methods to construct Steiner systems or other specialized designs from specific classes of ternary codes.

5. Lattice Applications:

   - Further investigate the unique properties of lattices derived from ternary codes for potential applications in cryptography and coding theory.
   - Explore the sphere-packing and covering properties of these lattices in more detail.

6. Decoding Algorithms:

   - Develop efficient decoding algorithms specifically tailored for ternary cyclic codes, leveraging their unique structural properties.

7. Quantum Error Correction:

   - Investigate the potential of ternary cyclic codes in quantum error correction, extending current binary approaches.

8. Cryptographic Applications:

   - Explore the use of ternary cyclic codes and their associated designs in developing new cryptographic primitives or improving existing ones.

9. Theoretical Bounds:

    - Work on tightening the bounds on minimum distance and other parameters for ternary cyclic codes.
    - Investigate the asymptotic behavior of these codes as length increases.

10. Automorphism Group Structures:

    - Conduct a deeper analysis of the automorphism groups of ternary cyclic codes and their relationship to code properties.

11. Weight Enumerator Polynomials:

    - Study the properties of weight enumerator polynomials for ternary cyclic codes and their connections to other mathematical objects.

12. Algebraic Geometry Codes:

    - Explore connections between ternary cyclic codes and algebraic geometry codes over ternary fields.

These directions for future research have the potential to further advance our understanding of ternary codes and their applications, as well as contribute to the broader fields of coding theory and discrete mathematics.

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APPENDICES

APPENDIX A: MAGMA CODE FOR GENERATING TERNARY CYCLIC CODES

This appendix provides the Magma code used to generate and analyze the linear cyclic ternary codes studied in this thesis.

0.1 Code Generation Functions

```
// Function to generate all cyclic codes of given length
GenerateTernaryCyclicCodes := function(n)
F := GF(3);
R<x> := PolynomialRing(F);
factors := Factorization(x^n - 1);

// Generate all possible generator polynomials
genPolys := [];
for subset in Subsets({1..#factors}) do
g := &*[factors[i][1]^factors[i][2] : i in subset];
Append(~genPolys, g);
end for;

// Create cyclic codes from generator polynomials
codes := [];
for g in genPolys do
C := CyclicCode(n, g);
Append(~codes, C);
end for;

return codes;
```

```
end function;
```

0.2 Weight Distribution Analysis

```
// Function to compute weight distribution
ComputeWeightDistribution := function(C)
weights := [];
for c in C do
Append(~weights, Weight(c));
end for;

// Count occurrences of each weight
dist := [];
for w in [0..Length(C)] do
count := #[x : x in weights | x eq w];
if count gt 0 then
Append(~dist, <w, count>);
end if;
end for;

return dist;
end function;
```

APPENDIX B: DESIGN CONSTRUCTION ALGORITHMS

This appendix details the algorithms used to construct combinatorial designs from the generated codes.

0.1 Assmus-Mattson Implementation

```
// Function to check if supports form t-design
CheckTDesign := function(C, t)
// Get codewords of each weight
byWeight := {};
for c in C do
w := Weight(c);
if w notin Keys(byWeight) then
byWeight[w] := {};
end if;
Include(~byWeight[w], Support(c));
end for;

// Check t-design properties
for w in Keys(byWeight) do
if IsDesign(byWeight[w], t) then
return true, w, byWeight[w];
end if;
end for;

return false, 0, {};
end function;
```

0.2 Kramer-Mesner Method

```
// Function to implement Kramer-Mesner construction
KramerMesnerConstruction := function(C, t, lambda)
// Get automorphism group
Aut := AutomorphismGroup(C);

// Compute orbits under group action
orbits := Orbits(Aut);

// Construct incidence matrix
M := KramerMesnerMatrix(orbits, t, lambda);

// Solve system to find designs
return SolveKramerMesner(M);
end function;
```

APPENDIX C: LATTICE CONSTRUCTION AND ANALYSIS

This appendix provides the mathematical details and algorithms for constructing and analyzing lattices from the ternary codes

0.1 Construction A Implementation

```
// Function to implement Construction A
ConstructLattice := function(C)
// Get code parameters
n := Length(C);
k := Dimension(C);

// Create lattice basis matrix
basis := [];
for c in Generators(C) do
Append(~basis, Vector(c));
end for;
for i in [1..n] do
v := ZeroVector(F, n);
v[i] := 3;
Append(~basis, v);
end for;

return LatticeWithBasis(Matrix(basis));
end function;
```

0.2 Lattice Properties Analysis

```
// Function to compute lattice properties
AnalyzeLattice := function(L)
```

```
props := rec<>;
props‘dimension := Dimension(L);
props‘determinant := Determinant(L);
props‘minNorm := MinimumNorm(L);
props‘kissingNumber := KissingNumber(L);
props‘theta := ThetaSeries(L, 5);
return props;
end function;
```