THE STRUCTURES OF MATRICES AND INDICES OF ZERO DIVISOR GRAPHS OF 3,4-RADICAL ZERO COMPLETELY PRIMARY FINITE RINGS

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A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Pure Mathematics of Masinde Muliro University of Science and Technology

September, 2025

DECLARATION

This thesis is my original work prepared with no other than the indicated sources and support and has not been presented elsewhere for a degree or any other award.

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DEDICATION

This work is entirely dedicated to my beloved wife, Betha and son, Casson and my parents, Mark and Joan.

ACKNOWLEDGEMENTS

I begin by giving special thanks to my creator, the Almighty God for granting me a gift of life and the mental strength to pursue this research. My sincere gratitude goes to my supervisors: Prof. Owino Maurice Oduor and Dr. Ojiema Michael Onyango for their unrelenting support towards the identification of gaps in the research area and their commitments towards the development of this work. The support you gave me was immense and overwhelming. You are my great mentors. More appreciation goes to Masinde Muliro University of Science and Technology for giving me an opportunity to study and carry out this research from the great institution. And to the Mathematics department of Masinde Muliro University of Science and Technology headed by Prof. Simiyu Achilles, I am so grateful for the scholarly advices I always received from you which propelled me to the completion of this work. To my beloved wife Betha, parents: Mark Ndago and Joan Atieno, I am thankful for the moral support and the motivation that you accorded me to proceed on with the studies. May God bless you all.

ABSTRACT

A zero divisor graph of the ring R is a graph whose vertices are entirely from the set of zero divisors of the ring and two vertices of the graph are adjacent if and only if their product is zero. The study of zero divisor graphs is important for it provides a better way of relating graph geometry to matrix conformations and formulation of encryption algorithms in coding therefore fundamental in interpretation of patterns, maps and networks in computer programs and modelling. Reasonable research has been done concerning zero divisor graphs of commutative rings with identity $1 \neq 0$, however the generalization of the structures of the matrices of zero divisor graphs is still not extensive in the existing literature. Much of the recent works on zero divisor graphs of finite commutative rings have been restricted to the algebraic properties of the graphs such as colouring, girth, spectral radii and classification in terms of their completeness up to isomorphism. This has left the characterization of finite commutative rings via the structures of the matrices and indices of their graphs fairly untouched. In particular, matrices and indices of the zero divisor graph $\Gamma(R)$ of finite commutative rings of 3-radical zero and 4-radical zero have not been characterized. This research has determined and investigated the properties of the matrices and indices of $\Gamma(R)$ of finite rings R with unique maximal ideal $J(R)$ such that $J(R)^3 = (0)$ and $J(R)^2 \neq (0)$; $J(R)^4 = (0)$ and $J(R)^3 \neq (0)$. It has also established the singularity and the relationship that exist between the eigenvalue multiplicities in the spectrum to the nullity of the graphs. We have validated the construction of these classes of rings using idealization procedure and the zero divisor graphs drawn from the isolated zero divisors using the Tikz Software. The matrices have been formulated from the graphs using standard definitions and the Mathematica software applied in investigating some of their algebraic properties. The results of this study can find an application to networking such as Google PageRank algorithms, developing more improved codes for better graph interpretation in operation systems. It will also advance the ring classification problem by revealing the interplay between ring theory, graph theory and linear algebra therefore contributing fundamentally to the literature of advanced algebra.

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ABBREVIATIONS AND NOTATIONS

R	-	A Ring.
$Z(R)$	-	Set of zero-divisors of R.
$(Z(R))^*$	-	Set of non zero zero-divisors of R.
$R/Z(R)$	-	The quotient ring R modulo $Z(R)$.
R^*	-	Group of units of R.
$J(R)$	-	Jacobson radical of R.
$\oplus$	-	Direct sum.
$A = [a_{ij}]$	-	Adjacency matrix associated with $\Gamma(R)$.
$L = [l_{ij}]$	-	Laplacian matrix associated with $\Gamma(R)$.
$[d_{ij}]$	-	Distance matrix associated with $\Gamma(R)$.
$d(v_i, v_j)$	-	Distance between the vertices v_i and v_j.
$M_n(R)$	-	Square matrix ring over R.
$GR(p^{kr}, p^k)$	-	Galois ring of order p^{kr} and characteristic p^k.
$G(R)$	-	The total graph.
$\Gamma(R)$	-	Anderson and Livingston's zero divisor graph (Zero divisor graph whose vertices are non-zero zero divisors).
$\mid \Gamma(R) \mid$	-	The number of vertices in $\Gamma(R)$.
$diam(\Gamma(R))$	-	Diameter of the graph $\Gamma(R)$.
$\omega(G(R))$	-	Clique number of the graph $G(R)$.
$\Gamma_E(R)$	-	Mulay's graph/Condensed zero divisor graph of R.
$\Gamma_I(R)$	-	Redmond's zero divisor graph.
$G(V, E)$	-	The undirected graph with the vertex set V and a set of edges E.
$\Delta(G)$	-	The maximum degree of graph G.
$\delta(G)$	-	Minimum degree of graph G.
$\sigma([A]), \sigma([L])$	-	Spectrum of the adjacency and Laplacian matrix respectively.
$V(\Gamma(R))$	-	The vertex set of $\Gamma(R)$.
$\mid V(\Gamma(R)) \mid$	-	The cardinality of the vertex set, $V(\Gamma(R))$.
$Aut(R)$	-	Automorphism group of the ring R.
$b(\Gamma(R))$	-	The binding number of the zero divisor graph $\Gamma(R)$.
$W(\Gamma(R))$	-	Wiener index of the zero divisor graph $\Gamma(R)$.
$Z_1(\Gamma(R))$	-	First Zagreb index of the zero divisor graph $\Gamma(R)$.
$Z_2(\Gamma(R))$	-	Second Zagreb index of the zero divisor graph $\Gamma(R)$.
$A(\Gamma(R))$	-	Average disorder number of the zero divisor graph $\Gamma(R)$.
$\mu(\Gamma(R))$	-	Average distance index of the Zero divisor graph $\Gamma(R)$.
$Dim(R)$	-	Dimension of the ring R.
$T(\Gamma(R))$	-	Simple topological index of $\Gamma(R)$.
$I(\Gamma(R))$	-	Inverse degree of $\Gamma(R)$.
S	-	A non-empty subset of $V(\Gamma(R))$.
$N(S)$	-	The neighbourhood of S.
$Char(R)$	-	Characteristic of a ring R.
$Ann_R v$	-	Annihilator of v in R.
$\mathbb{Z}_{p^n}[i]$	-	Gaussian ring of integers modulo p^n.
$N_G(v)$	-	The neighbourhood of a vertex v in Graph G.
$M_{s \times s}(\mathbb{F})$	-	A square matrix Ring of order s over the field $\mathbb{F}$.

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

A completely primary finite ring is a ring R with identity $1 \neq 0$ whose subset of zero divisors form a unique maximal ideal $J(R)$ [57]. The concept of completely primary finite rings has been an active area of research for a while. Raghavendran in [57] provided a foundation in the interpretation of the Jacobson radical $J(R)$ in terms of the zero divisors $Z(R)$, the order of R and its group of units R^*.

A number of studies have been advanced on these classes of rings. For instance, Chikunji in [14] provided a review on the theory of completely primary finite rings and to a greater extent, a class of finite rings whose set of zero divisors forms an additive group. Further, a description was provided on their structures by outlining a general structural representation as an additive direct sum of cyclic modules over their maximal Galois subring. In [9], Chikunji further classified cube radical zero completely primary finite rings R such that if $J(R)$ is the Jacobson radical of R then, $(J(R))^3 = (0)$ and $(J(R))^2 \neq (0)$. The problem of enumerating these rings in the particular case where the maximal Galois subrings lie in the center by way of giving a method of determining the isomorphism classes was considered.

Owino and Ojiema in [48] did an important research to characterize the unit groups of some classes of power four radical zero commutative completely primary finite rings R in which if $J(R)$ is the Jacobson radical, $(J(R))^4 = (0)$ and $(J(R))^3 \neq (0)$. They further determined the structures of their unit groups.

A graph is a representation of a set of objects where some pairs of the objects are connected by links [53]. For the zero divisor graph, the set of objects are the zero divisors of the ring R. A lot of research has been done on zero divisor graphs and unit groups of completely primary finite rings with great success as seen in the next section on the reviewed literature. Much study has been done on the zero divisor graphs for commutative rings with identity. Few expositions have been given by Anderson and Livingston [3], Beck [5], Mulay [38] and Redmond [59] respectively. For instance, Beck in [5] constructed a graph of zero divisors in which every element is a vertex and focused on the determination of the coloring of the graphs, i.e obtaining the chromatic number of the graphs. Anderson and Livingston in [3] redefined the concept of the zero divisor graph and constructed graphs in which every vertex is a nonzero-zero divisor. Their motivation was to give a better illustration of the zero divisor structure of the ring. Mulay in [38] introduced another zero divisor graph which is constructed from the classes of zero divisors determined by annihilator ideals rather than the individual zero divisors themselves. A further study was done by Redmond in [59] who introduced another zero divisor graph in which an element $x \in R \setminus I$ is a vertex when I is an ideal of R and $x, y \in R \backslash I$ are adjacent when $x.y = 0$. The relationship between the zero divisor graphs $\Gamma_I(R)$ and $\Gamma(R/I)$ was discussed. The findings, however, did not however extend to the matrices of the said graphs.

Various results have been obtained on matrices of finite rings in which the product of any two zero divisors is zero. Determination of properties of matrices of zero divisor graphs of classes of 3-radical zero and 4-radical zero finite rings has not received much attention. Despite the fact that studies on the matrices for the classes of rings in question have not been done, some findings exist on other classes of rings. For instance, a study on the adjacency matrix for zero divisor graphs over finite rings of Gaussian integer modulo n was done by Pranjali, Amit and Vats in [54]. They characterized the order, singularity, trace of the adjacency matrix of $\mathbb{Z}_{p^n}[i]$ for $p \equiv 1(mod\ 4)$ and $p \equiv 3(mod\ 4)$. They extended the study to the analysis of the adjacency matrix and the neighbourhood associated with the zero divisor graph of finite commutative rings in [53]. The study concentrated on commutative rings of direct products of the form $R = R_1 \times R_2$ where R_1 and R_2 are integral domains with variations of the values of $p = 2, 3, 5$ for $R = \mathbb{Z}_p \times \mathbb{Z}_p$. The authors determined the rank, eigenvalues, symmetry and determinant of the neighbourhood associated with the graph for $Z(R)$.

In [42], Ndago, Ojiema and Owino obtained the adjacency and incidence matrices of the zero divisor graph $\Gamma(R)$ of a class of the square radical zero completely primary finite rings. The research further characterized the matrix algebraic properties of the class of completely primary finite rings with characteristics p and p^2. This study did not however extend to the higher indices of nilpotence of $Z(R)$ for finite rings. Moreover, no result was presented with regards to graph indices of the aforementioned classes of rings. In this research, we have given a special focus to adjacency, Laplacian and distance matrices due to the display of similarity in structural construction and algebraic interpretation in properties for the classes of rings studied.

Characterization of rings have also been done via the Adjacency, Laplacian and distance matrices of their zero divisor graphs. The adjacency matrix of a zero divisor graph $\Gamma(R)$ is defined by
$[A]$, where

$$a_{ij} = \begin{cases} 1, & x_i x_j = 0, i \neq j; \\ 0, & \text{otherwise} \end{cases}$$

for any vertices x_i, x_j and for all $i, j \in \mathbb{N}$. Given an adjacency matrix [A] and a degree matrix [D] which is the diagonal matrix whose diagonal entries are the degrees of vertices of $\Gamma(R)$ and 0-elsewhere, the Laplacian matrix is an $n \times n$ matrix $[L] = [D] - [A]$. Further, the distance between two vertices v_i and v_j is denoted by $d(v_i, v_j)$ and it is defined as the length of the shortest path between vertices v_i and v_j [60]. The distance matrix of a graph G having n vertices is a symmetric matrix $[d_{ij}]$ whose entry d_{ij} is defined as

$$d_{ij} = \begin{cases} d(v_i, v_j), & \text{if } i \neq j; \\ 0, & \text{if } i = j \text{ for } 1 \leq i, j \leq n \in \mathbb{N}. \end{cases}$$

Properties of the adjacency matrices of zero divisor graphs have been conducted, evident in [53, 54]. In the papers, the authors concentrated on the analysis of the adjacency matrix and the neighbourhood $N_G(v)$, of a vertex v associated with the zero divisor graph of finite commutative rings of direct product such that $R = R_1 \times R_2$ and the adjacency matrix for zero-divisor graphs over finite rings of Gaussian integers. This research concentrated on the characterization of such matrices corresponding to the rings of Gaussian integers modulo n. The study evaluated the number of zero divisors in each case and examined the orientation of their matrix and further generalized the order, rank, determinants and the eigenvalues of the matrices in each case.

In [56], an investigation on the Laplacian eigenvalues of the zero divisor graph associated to the ring of integers modulo n was performed. Here, it was noticed that the structure and Laplacian spectrum of $\Gamma(\mathbb{Z})$ for $n = p^{N_1}q^{N_2}$ where $p < q$ are primes and both N_1 and N_2 are positive integers. Further, it was demonstrated that the Euler's totient function ϕ satisfies $\phi(pq) = \phi(p)\phi(q)$ where p, q are relatively prime. Further investigations on the Laplacian matrices can also be observed in [51] and [61] which were based on signless Laplacian spectrum of zero divisor graphs of the ring $\mathbb{Z}_n$, and Laplacian eigenvalues of the zero divisor graphs of the ring $\mathbb{Z}_n$, respectively.

Other than giving an analysis of the matrix properties of the zero divisor graphs of the classes of rings, this study has also explored the graph indices of $\Gamma(R)$ such as the Wiener number, binding number and the average disorder number together with bounds on both first and second Zagreb indices of the graph $\Gamma(R)$. The Wiener index denoted as W and also known as path number or the Wiener number is a graph index on a graph with n nodes and is defined by

$$W = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}$$

where d_{ij} is the shortest distance between any two vertices v_i and v_j [52]. The Wiener index $W(G)$ of a graph G with vertex count $\mid G \mid$ is related to the average disorder number of the graph $A(G) = \frac{2W(G)}{|G|}$ and the average distance $\mu(G)$ between the vertices of G as demonstrated in [22] is

$$\mu(G) = \frac{W(G)}{\binom{\mid V(G) \mid}{2}}.$$

The first Zagreb index is the sum of the squares of degrees of the vertices and the second Zagreb index is the sum of the products of the degrees of the pairs of adjacent vertices. We denote the first and second Zagreb indices of $\Gamma(R)$ as $Z_1(\Gamma(R))$ and $Z_2(\Gamma(R))$ respectively. Therefore for any $v_i, v_j \in V(\Gamma(R))$,

$$Z_1(\Gamma(R)) = \sum_{i=1}^{n} (deg(v_i))^2$$

and

$$Z_2(\Gamma(R)) = \sum_{i,j=1}^{n} (deg(v_i))((deg(v_j))$$

where $i \neq j$. The Zagreb indices were introduced in [25] and given an elaboration in [26]. Fundamental properties of the indices were given a summary in [43].

This research has revealed a representation of zero divisor graphs via their matrices, analysed the geometric properties of the graphs and matrix algebraic properties together with the indices of the classes of 3-radical zero and 4-radical zero completely primary finite rings.

1.2 Basic Concepts

This section is devoted to some important definitions and results that have been used in the subsequent chapters. It is not intended to be exhaustive but instead, a collection of results which will play important role in the sequel.

1.2.1 Ring Theoretic Concepts

The definitions in this subsection can be obtained in any abstract algebra text, for example [35].

Definition 1.2.1. *A commutative ring R with identity* 1 *is a non-empty set endowed with two binary operations, addition and multiplication such that;*

(i) R is an additive Abelian group,

(ii) R is a multiplicative semigroup,

(iii) Multiplication is commutative, that is, $xy = yx$, for all $x, y \in R$,

(iv) Multiplication distributes over addition; $x(y + z) = xy + yz$ and
$(y + z)x = yx + zx$, forall $x, y, z \in R$, and,

(v) $1.x = x.1 = x$ for all $x \in R$.

Definition 1.2.2. *Two non-zero elements a and b of a ring are respectively called a left zero divisor and right zero divisor if $ab = 0$. An element that is both a left and a right zero divisor is called a two-sided zero divisor. If the ring is commutative then the left and right zero divisors are the same.*

Theorem 1.2.1. *([16]) If a ring R is finite then every left unit is a right unit and every left zero divisor is a right zero divisor. Furthermore, every element of R is either a zero divisor or a unit.*

Theorem 1.2.2. *([16]) If a ring R has $n \geq 2$ left zero divisors (including zero), then R is a finite ring and $|R| \leq n^2$.*

Definition 1.2.3. *An* Ideal *in a commutative ring R is a subset I of R such that*

(i) $0 \in I$,

(ii) $x, y \in I$ implies that $x + y \in I$, and

(iii) $x \in I$ *and* $r \in R$ *then* $rx \in I$.

Definition 1.2.4. *The Nilradical* $N(R)$ *of a ring* R *is the set* $\{x \in R : x^n = 0\}$ *for some positive integer* n.

Definition 1.2.5. *Let* R *be a commutative ring with identity,* $Z(R)$ *its set of zero divisors and* $Nil(R)$ *its ideal of nilpotent elements. The zero divisor graph* $\Gamma(R)$ *of* R *is the graph of* $Z(R)\backslash\{0\}$, *with distinct vertices* x *and* y *adjacent if and only if* $xy = 0$.

Definition 1.2.6. *If* R *is a ring and* M *and* N *are* R*-modules, then a function* $f : M \longrightarrow N$ *is an* R*-homomorphism (or* R*-map) if, for all* $m, m' \in M$ *and all* $r \in R$,

(i) $f(m + m') = f(m) + f(m')$, *and*

(ii)$f(rm) = rf(m)$.

Definition 1.2.7. *An Ideal* I *of a ring* R *is said to be maximal if;*

(i) $I \neq R$, *and*

(ii) Given an ideal $J \supsetneq I$ *then* $J = R$.

Definition 1.2.8. A completely primary finite ring *is a ring* R *with identity* $1 \neq 0$ *whose subset of all the zero divisors form a unique maximal ideal* J.

Definition 1.2.9. *Let* $\mathbb{Z}$ *denote the ring of all integers,* x *an indeterminate,* p *a prime and* n, r *arbitrary positive integers. Let* $f(x) = a_0 + a_1x + ... + a_{n-1}x^{r-1} + x^r \in \mathbb{Z}_{p^k}[x]$, *be a monic irreducible polynomial of degree* r *whose image in* $\mathbb{Z}_p$ *is irreducible. Suppose* $(f(x))$ *denotes the ideal generated by* $f(x)$ *in* $\mathbb{Z}_{p^k}[x]$, *then the quotient ring* $R_\circ = \mathbb{Z}_{p^k}[x]/(f(x))$ *is called the Galois ring of order* p^{kr} *and characteristic* p^k *denoted by* $GR(p^{kr}, p^k)$.

Proposition 1.2.1. *([13]) Let* R *be a completely primary finite ring. Then the group of units,* R^* *of* R *contains a cyclic subgroup* $< b >$ *of order* $p^r - 1$ *and* R^* *is semi direct product of* $1 + J$ *and* $< b >$.

Corollary 1.2.1. *([16]) Let R be a finite ring with identity $1 \neq 0$. Then every non-trivial ideal of R consists entirely of zero divisors.*

1.2.2 Graph Theoretic Concepts

A good reference for the definitions used in Graph theory is [19].

Definition 1.2.10. *A* graph *$G = (V, E)$ is a pair consisting of V, a set of nodes or points or vertices and E, a set of lines or arcs or edges joining the vertices. Two vertices are said to be adjacent if they are joined by an* edge*. An edge from a vertex to itself is a* loop.

Definition 1.2.11. *A* zero divisor graph *of a ring R is a graph $\Gamma(R)$ with vertices from the set $Z(R)$ of zero divisors and two vertices are adjacent if $xy = 0$.*

Definition 1.2.12. *A* simple *graph is a graph with no loops on a vertex and no multiple edges between a pair of vertices.*

Definition 1.2.13. *A graph is* connected *if any pair of vertices are joined by at least one path, otherwise the graph is disconnected.*

Definition 1.2.14. *Let G be undirected graph with vertex set V. The degree of vertex $v \in V$ is the number of vertices that are adjacent to v. It is denoted by $deg(v)$.*

Theorem 1.2.1. *[30] If G is an undirected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_1, e_2, \cdots, e_m\}$, then $\sum_{i=1}^{n} deg(v_i) = 2m$.*

Definition 1.2.15. *The* diameter *of a connected graph G is the supremum of the distances between any two vertices. It is denoted by $diam(G)$.*

Theorem 1.2.2. *[63] In any connected graph, $rad(G) \leq diam(G) \leq 2rad(G)$.*

Definition 1.2.16. *Let v_1, v_2 be two vertices in a graph G, the* distance *between the vertices is the length of the shortest path from v_1 and v_2 in G denoted as $d(v_1, v_2)$. If G is disconnected and v_1 and v_2 in different components, then $d(v_1, v_2) = \infty$.*

Theorem 1.2.3. *[2] Let R be a finite ring such that $\Gamma(R)$ is $r-$partite.*

(i) Then r is a prime power.

(ii) If $r \geq 3$, then at most one partitioning subset of the vertices of $\Gamma(R)$ can have more than one vertex.

(iii) If R is reduced, then $\Gamma(R)$ is bipartite (i.e., $r = 2$).

(iv) If R is reduced and $\Gamma(R)$ is bipartite, then $\Gamma(R)$ is complete bipartite.

Definition 1.2.17. *The* binding number *of $\Gamma(R)$ denoted by $b(\Gamma(R)) = \frac{|N(S)|}{|S|}$ where $S \subseteq V(\Gamma(R)), N(S) \neq V(\Gamma(R)), S \neq \emptyset$ such that*

(i) $N(S) \cup S = V(\Gamma(R))$,

(ii) $N(S) \cap S = \emptyset$,

(iii) $deg(u) \leq deg(v)$ for all $u \in S, v \in N(S)$, and

(iv) no pair of vertices in S are adjacent.

1.3 Statement of the Problem

The zero divisor graphs of finite rings have been researched on by many authors. However, the characterization of finite commutative rings via matrices and indices of their graphs for classes of 3-radical zero and 4-radical zero completely primary rings have not been done. Given a square matrix M_n of order n, it is not generally known the classes of finite rings for which it is a representation of the zero divisors. There exist some little expositions on the matrices of Galois rings, square radical zero completely primary finite commutative rings and other classes of rings of integer modulo n. For a given finite ring R with the unique maximal ideal $J(R)$ such that $(J(R))^n = (0), (J(R))^{n-1} \neq (0)$ and $R/J \cong GF(p^r)$ for $r \geq 1$ and $n \in \mathbb{N}$, the general structures of the matrices and indices of their zero divisor graphs are still unknown. This research reviews aspects of graph geometric properties, determines and investigates the matrices of $\Gamma(R)$ of finite rings R with the unique maximal ideal $J(R)$ such that $(J(R))^3 = (0), (J(R))^2 \neq (0)$ and $(J(R))^4 = (0), (J(R))^3 \neq (0)$ with an aim of classifying them by characterizing their algebraic properties. We also analyse some indices of $\Gamma(R)$ such as the Wiener index and its invariants such as the average disorder number and the average distance index of the graph and further investigate the bounds on the Zagreb indices of $\Gamma(R)$ with respect to the maximum and minimum degrees of $\Gamma(R)$.

1.4 Objectives of Study

1.4.1 Main Objective

The main objective of this study is to investigate the structures of matrices and indices of zero divisor graphs of classes of 3-radical zero and 4-radical zero completely primary finite rings.

1.4.2 Specific Objectives

The specific objectives of this research are;

(i) To characterize the structures of zero divisors and the graphs of 3-radical zero and 4-radical zero commutative completely primary finite rings.

(ii) To determine the matrices associated with the zero divisor graphs of 3-radical zero and 4-radical zero commutative completely primary finite rings.

(iii) To investigate the Wiener index, average disorder number index, distance index and bounds on the zero divisor graph of the classes of 3-radical zero and 4-radical zero completely primary finite rings.

1.5 Methods of Study

The following methods have been used in this research;

(i) The method of Idealization of R'-modules based on Raghavendran's principles to perform the construction of 3-radical zero and 4-radical zero completely primary finite rings.

This method transforms a module over a commutative ring R into an ideal similar to R [39]. As demonstrated in [32], if R is a commutative ring and M is a module in R, then $R \times M$ can possibly yield a ring structure by giving it addition and multiplication operations as

$$(r, a) + (s, b) = (r + s, a + b)$$

and

$$(r, a)(s, b) = (rs, rb + sa).$$

By associating M with $\{0\} \times M$, this construction transforms M from a module over R to a ring ideal similar to R. The ring formed is identified as $R(+)M$ which is said to have been constructed through idealization of M over R.

Together with idealization in this method, we have applied the theorem developed by Raghavendran (See Theorem 2.2.1) in obtaining the structures of the ideal $Z(R)$ and its order for every class of completely primary finite ring discussed.

(ii) Tikz software to develop codes describing a Cartesian graphical orientation has been used to draw the zero divisor graphs. This has been achieved through designing algorithms which generate position of the nodes, their orientation on the axes and the fill on each node. From the multiplication obtained in (i), an algorithm describing link pattern based on the existence of an edge between pair of vertices has been formulated to complete the connectedness of the graphs. Finally, we have generated a code to label a vertex (zero divisor) on each node in the graph.

(iii) Standard definitions to construct Adjacency, Laplacian and distance matrices from the graphs drawn in method (ii). We describe the standard definitions we have used to construct the matrices from the zero divisor graphs of the classes of rings as follows.

The adjacency matrix of a zero divisor graph $\Gamma(R)$ is defined as: $[A]$, where

$$a_{ij} = \begin{cases} 1, & x_i x_j = 0; \\ 0, & \text{otherwise} \end{cases}$$

for any vertices x_i, x_j and for all $i, j \in \mathbb{N}$.

Given an Adjacency matrix $[A]$ and a degree matrix [D] which is the diagonal matrix whose diagonal entries are the degrees of vertices of $\Gamma(R)$ and 0-elsewhere, the Laplacian matrix is an $n \times n$ matrix $[L] = [D] - [A]$.

The distance between two vertices v_i and v_j is denoted by $d(v_i, v_j)$ and it is defined as the length of the shortest path between vertices v_i and v_j [60]. The distance matrix of a graph G having n vertices is a symmetric matrix $[d_{ij}]$ whose entry d_{ij} is defined as

$$d_{ij} = \begin{cases} d(v_i, v_j), & \text{if } i \neq j; \\ 0, & \text{if } i = j \text{ for } 1 \leq i, j \leq n. \end{cases}$$

(iv) We have used Linear Algebra techniques of handling matrices and the MATHEMATICA software to analyse properties of the matrices such as determinant, trace, eigenvalues and the characteristic equations.

1.6 Significance of the Study

Graphs are useful in generating patterns and modelling in computer applications. This study obtains simple graphs (Zero divisor graphs) of the classes of 3-radical zero and 4-radical zero completely primary finite rings which make interpretation of patterns in computer modelling simpler. These graphs are important for networking in communication in protocol designs and determination of page links, through vertices as the pages and vertex degrees as the links associated to every page. The results of this study will also advance the ring classification problem by revealing the interplay between ring theory, graph theory, and linear algebra. This is achieved by interpreting the structures of the zero divisors of the ring through the order of the graph, diameter, degree of vertices and formulating the matrices from the graphs. Matrices are then analysed through linear algebra techniques in obtaining the rank, determinant, trace, order and the eigenvalues. This therefore contributes fundamentally to the literature of advanced algebra. This study is also an extension of the studies done on characterization of finite commutative rings.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Let R be a finite ring with identity $1 \neq 0$ and $J(R)$ be the Jacobson radical of R. We say that R is completely primary if $J(R)$ is its unique maximal ideal and $(J(R))^n = (0), n \in \mathbb{N}$. In this chapter, we take a survey of some important studies done on completely primary finite rings , graphs of finite rings and the structures of their zero divisors, their units and automorphisms of their unit groups. We also review some existing literature on the Wiener index and its invariants on some classes of graphs and matrices of certain classes of finite rings.

2.2 Completely Primary Finite Rings

The concept of completely primary finite rings has been an active area of research for quite sometime. In this section, important leading findings are given more attention since they form the basis of this research. Raghavendran's study in [57] focussed on the finite associative rings where the structures of prime-power rings under various conditions were determined. The study began by considering a set $\Re$ which is simultaneously left vector space and a right vector space over the same field P subjected to the condition that $a(xb) = (ax)b$ for all $a, b \in \Re$. A generalization of the Theorem of Wedderburn on finite division rings with a finite number of elements was also given. He obtained some properties of completely primary finite rings. For instance, the following important result was obtained.

Theorem 2.2.1. *(Raghavendran[57]) Let R be a finite ring with multiplicative identity $1 \neq 0$, whose zero divisors form an additive group $J(R)$. Then,*

(i) $J(R)$ is the Jacobson radical of R,

(ii) $\mid R \mid= p^{nr}$, *and* $\mid J(R) \mid= p^{(n-1)r}$ *for some prime integer* p *and some positive integers* n *and* r*,*

(iii) $(J(R))^n = (0)$.

In the same paper [57], gave a foundation on the orders of the Jacobson's radical which is the set of all the zero divisors in relation to the order of the finite associative ring R. This theorem has been used by other scholars in characterizing the structures of units and zero divisors for square radical and cube radical zero completely primary finite rings. The research in [57] however did not extend to the zero divisor graphs of such ring R and by extension, matrices of their zero divisor graphs.

Much of the recent work on completely primary finite rings demonstrates the great and fundamental importance of these rings in the structure theory of finite rings with identity. More evidence is clear from the work done by Chikunji in [14] on a review of the theory of completely primary finite rings to a greater extent, a class of finite rings whose set of zero divisors form an additive group. Chikunji further described the structures of such rings and provided a general representation for these rings as additive direct sum of cyclic modules over their maximal Galois subring. A review of the properties of completely primary finite rings and their representation with respect to the Galois ring became a subject. The following was a consequence from the Representation Theorem.

Theorem 2.2.2. *(see Theorem 4.8 in [14]) Let* R *be a completely primary finite ring of order* p^{nr} *with unique maximal ideal* J *such that* $\mid R/J \mid= p^r, Char(R) = p^k$ *for some positive integers* n, r *and* k*. If* $R_\circ$ *is the maximal Galois subring of* R*, then there exist* $x_1, ..., x_h \in J$ *and* $\sigma_1, ..., \sigma_h \in Aut(R_\circ)$ *such that* $R = R_\circ \oplus R_\circ x_i \oplus, ... \oplus R_\circ x_n$ *and* $x_i r = r^{\sigma_i} x_i$ *for every* $r \in R_\circ$ *and each* $i = 1, 2, ..., h$*.*

Moreover, the automorphisms $\sigma_1, ..., \sigma_h \in Aut(R_\circ)$ are uniquely determined by $R_\circ$ and R. This result is limited to representation of R as an additive direct sums of cyclic modules over their maximal Galois subring.

Chikunji in [12] did a study on automorphisms of completely primary finite rings of characteristic p such that if $J(R)$ is the Jacobson radical of R then $(J(R))^3 = (0), (J(R))^2 \neq (0)$ and the annihilator of $J(R)$ coincides with $(J(R))^2$ and $R/J(R) \cong GF(p^r)$. In his work, the main result is outlined in the following theorem.

Theorem 2.2.3. *(see Theorem 3.12 in [12]) If $\mathbb{F}$ is the Galois field $GF(p^r)$ and $1 \leq t \leq s^2$ for a fixed s and $t-$ dimensional $\mathbb{F}$-spaces U and V respectively and $(a_{ij}^k) \in \mathbb{M}_{s\times s}(\mathbb{F})$ are t, linearly independent matrices. Then $Aut(R) \cong [\mathbb{M}_{t\times s}(\mathbb{F}) \times (U \oplus V)] \times_{\theta_2} [Aut(\mathbb{F}) \times_{\theta_1} (GL(s,\mathbb{F}) \times GL(t,\mathbb{F}))]$ where $\{\theta_1, ..., \theta_t\}$ is the set of automorphisms of $\mathbb{F}$.*

In this result, the automorphism of the ring R was expressed as a product of matrix endomorphism over a Galois field. It is evident from [12] that there is a significant gap to be bridged in advancing the research on completely primary finite rings more so, looking at higher indices of nilpotence and extending the study to the matrix conformations of the graphs.

Another significant work worth mentioning was done by Ojiema *et al* in [44] on the automorphisms of the unit groups of the square radical zero finite commutative rings. They had a focus on rings of characteristics p and p^2 and characterized the unit group R^* of these rings by determining the structure of $R^* = \mathbb{Z}_{p^r-1} \times_\theta (1+J)$. They obtained the following result.

Theorem 2.2.4. *(see Theorem 4 in [44]) Let R^* be the unit group of a class of finite rings such that $Char(R) = p$ and p^2 then,*

(i) For $Char(R) = p$, we have $Aut(R^*) = Aut(\mathbb{Z}_{p^r-1}) \times Aut(B_p)$ and

$$| Aut(R^*) | = | \mathbb{Z}_{p^r-1} | \times | Aut(B_p) | = \Phi(p^r - 1). \prod_{i=0}^{(hr)-1} (p^{hr} - p^i).$$

(ii) When the characteristic of $R = p^2$, *then* $\mid Aut(R^*) \mid = \Phi(p^r). \prod_{i=0}^{((h+1)r)-1}(p^{(h+1)r} - p^i)$ *where* B_p *is a subring of endomorphism ring such that* $B_p = 1 + J$.

From this result, it follows that the order of the automorphism of the unit group R^* can be calculated as a product of the order of cyclic group with automorphism order of a subring of endomorphism ring B_p.

Owino in [46] determined the automorphisms of a class of completely primary finite rings $R = R' \oplus U$ where R' is the Galois ring of the form $GR(p^{nr}, p^n)$ for $n = 1, 2$ and $n \geq 3$ and every element of R being uniquely determined as $\alpha_\circ + \alpha_1 p + \cdots + \alpha_{n-1}p^{n-1} + U$ with $\alpha_\circ, \alpha_1, ..., \alpha_{n-1} \in F' = R'/pR'$. This research characterized the automorphisms of completely primary finite rings though the structure of the zero divisors was not attended to.

An extension of the above research was done by Owino *et al* in [47] where study on the units of completely primary finite rings of characteristic p^n was performed and a group of units of completely primary finite rings of characteristic p^n for some prime integer p and positive integer n was characterized. These rings satisfied $Z(R) = pR' \oplus U$, $(Z(R))^{n-1} = p^{n-1}R'$ and $(Z(R))^n = (0)$. The following result was among the major results established;

Theorem 2.2.5. *(see Theorem 1 in [47]) The unit group* R^* *of commutative completely primary finite ring* R *of characteristic* p^n *with the maximal ideal* $J(R)$ *such that* $(J(R))^2 = (0)$ *when* $n = 1, 2$ *and* $(J(R))^n = (0), (J(R))^{n-1} \neq (0)$ *when* $n \geq 3$ *and with invariants p(prime integer),* $p \in J(R), r \geq 1$ *and* $n \geq 1$ *is a direct product of cyclic groups as follows;*

(i) If $Char(R) = p$, *then* $R^* \cong \mathbb{Z}_{p^{r-1}} \times (\mathbb{Z}_p^r)^h$.

(ii) If $Char(R) = p^h$, *then* $R^* \cong \mathbb{Z}_{p^r-1} \times \mathbb{Z}_p^r \times (\mathbb{Z}_p^r)^h$.

(iii) If $Char(R) = p^n, n \geq 3$, then $R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^{n-1}}^{r-1} \times (\mathbb{Z}_2^r)^h, & \text{if } p=2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^{n-1}}^r \times (\mathbb{Z}_p^r)^h, & \text{if } p \neq 2. \end{cases}$

Where h is the dimension of R.

The theorem characterized the unit groups of finite rings of characteristic p^n as isomorphic to direct product of cyclic groups. This result is important however, it is only restricted to the unit groups of completely primary finite rings for $Char(R) = p^n$ where $n = 1, 2, 3, \cdots$

Some classifications have also been done on completely primary finite rings of 3-radical zero and 4-radical zero. For instance, Chikunji in [9] did a classification of cube radical zero completely primary finite rings R such that if $J(R)$ is the Jacobson radical of R then, $(J(R))^3 = (0)$ and $(J(R))^2 \neq (0)$ where the consideration was on the problem of enumerating these rings in the particular case where the maximal Galois subrings lie in the center by way of giving a method of determining the isomorphism classes. It was established that the number of mutually non-isomorphic rings having the property that $(J(R))^3 = (0)$, $(J(R))^2 \neq (0)$ with invariants p, n, r, s, t, d such that $t = s^2, 1+t = s^2, d+t = s^2$, or $1+t+d = (1+s)^2$ is equal to $r =\mid Aut(F) \mid = \mid Aut(R_\circ) \mid$ for every characteristic p, p^2 or p^3.

Owino and Ojiema in [48] characterized the unit groups of some classes of power four radical zero commutative completely primary finite rings R in which if $J(R)$ is the Jacobson radical then $(J(R))^4 = (0)$ and $(J(R))^3 \neq (0)$. This research was a breakthrough in the construction of the new class of completely primary finite rings as it also went further to determine the structures of their unit groups. This is presented in the following theorem.

Proposition 2.2.1. *(see Proposition 9 in [48]) If R is a class of power four radical zero completely primary finite ring with the Jacobson radical J, then its group of units is isomorphic to the cyclic groups characterized as follows for $r, s \in \mathbb{Z}^+$.*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8^{r-1} \times (\mathbb{Z}_2^r)^s, & \textit{if } p = 2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_p^r)^s, & \textit{if } p \neq 2 \end{cases}$$

From the result, the unit groups of a class of power four radical zero completely primary finite rings was found to be isomorphic to the direct product of cyclic groups of different orders when $p = 2$ and $p \neq 2$.

From the selected studies done on the completely primary finite rings, it can be noticed that a lot of work has been done on the characterization of such rings up to automorphism. So much has been done on the cube radical zero completely primary finite rings in terms of characterization of their unit groups. Not a lot is evident in power four radical zero finite completely primary rings a part from the cyclic groups.

2.3 Zero Divisor Graphs

The concept of zero divisor graphs of commutative rings was introduced by Ivstan Beck in [5]. The main objective was to determine the chromatic number of the graphs. This graph was denoted as $G(R)$ where R is the ring. From the graph, every element of R is a vertex and two vertices u and v are adjacent when $u.v = 0$. Since the main interest of Beck's work was the chromatic number of $G(R)$ denoted by $\chi(G(R))$, it was established that $\chi(G(R)) = \omega(G(R))$, where $\omega(G(R)$ is the clique number of R. The study did not however identify the matrix conformations and indices on $G(R)$.

In [3], the concept of zero divisors was redefined by Anderson and Livingston. Their notation for this graph was $\Gamma(R)$. In this graph, a vertex is a non zero-zero divisor. The graph has been considered to be better in terms of characterization of the zero divisors since it is not as 'noisy' as the total Beck's graph $G(R)$. The motivation in this research was to give a better illustration of the zero divisor structure of the ring. In this new zero divisor graph which is simple graph with edges defined the same way as in [5], that is, the vertices u and v are adjacent if the product $u.v = 0$, only the zero divisors of the ring are included (non-zero elements) and that $Ann_R v \neq (0)$ for all $v \in R$.

Afterwards, Mulay in [38] introduced another zero divisor graph which is constructed from classes of zero divisors determined by annihilator ideals rather than individual zero divisors themselves. The notation for this graph is $\Gamma_E(R)$. So, $u \sim v$ if $Ann_R u = Ann_R v$. The annihilator of u is the set of elements in the vertex set V such that for every $v \in V, uv = 0$. In graph theory terms, it is the set of vertices adjacent to u in $\Gamma_E(R)$ and, if there is a loop at u (meaning $u^2 = 0$), then u is self annihilating.

A similar study was done by Anderson and Livingston in [3] and Mulay in [38] respectively which associated a simple graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of non zero-zero divisors of R. This gave all possible zero divisor graphs $\Gamma(R)$ with the order, $\mid \Gamma(R) \mid \leq 4$ and established that $\Gamma(R)$ is finite if and only if R is either finite or R is an integral domain. In particular, if $1 \leq \mid \Gamma(R) \mid < \infty$, then R is finite with $\mid R \mid \leq \mid Z(R) \mid^2$ where R is not a field. Further, $\Gamma(R)$ is connected $diam(\Gamma(R)) \leq 3$ and if $\Gamma(R)$ contains a cycle, then the girth of $\Gamma(R)$ is less than 4, where the girth is the length of the smallest polygon starting and ending with the same vertex in $\Gamma(R)$.

Bozic and Petrovic in [7] studied zero divisor graphs of matrices over commutative rings. Their intention was to investigate the properties of directed zero divisor graphs of matrix rings. They used their results to discuss the relation between the diameter of the zero divisor graph of a commutative ring R and that of the matrix ring $M_n(R)$. Their study characterized the possible diameters of $\Gamma(M_n(R))$ in terms of the diameters of $\Gamma(R)$. It was discovered that $\Gamma(M_n(R))$ is connected and $diam(\Gamma(M_n(R))) \leq 3$. On the other hand, it was noticed that if R is a commutative ring such that $Z(R)^* \neq (0)$, then $diam(\Gamma(R)) \leq \ diam(\Gamma(M_n(R)))$. Other results obtained were that, for a commutative ring R, if every finite set of zero divisors from R has a nonzero annihilator then the $diam(\Gamma(M_n(R))) = 2$ and that if $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $diam(\Gamma(R)) = 1$ then $diam(\Gamma(M_n(R))) = 2$. The following theorem was established to show the relationship between the zero divisor graphs of total quotient ring $\Gamma(M_n(T(R)))$ and the zero divisor graphs $\Gamma(M_n(R))$ over matrix ring $M_n(R)$.

Theorem 2.3.1. *(see Theorem 3.2 in [7]) If $T(R)$ is the total quotient ring of R then* $\Gamma(M_n(R)) \cong \Gamma(M_n(T(R)))$.

These properties set a basis and give a projection on how the number of vertices of a zero divisor graph in a particular ring can show correspondence to the rank of the matrices of the graph. The rings discussed here were not completely primary finite rings.

Some other research have also been done concerning condensed versions. For instance, Florida and Sandra in [21] studied condensed zero divisor graphs (those whose vertices are equivalence classes of zero divisors of a ring R having exactly 5 vertices). They determined the graph with exactly 5 vertices which can be realised as condensed zero divisor graph of a ring. Here, the condensed zero divisor of a ring R, denoted as $\Gamma_E(R)$ is the graph associated with R whose vertices are classes of zero divisors whose pair of distinct classes $[r], [s]$ are adjacent if and only if $[r].[s] = 0$ where $[r].[s] = [rs]$. Their study established that if two points in the condensed zero divisor graph are adjacent to the same set of vertices but are not adjacent to one another, then at least one is self-annihilating. Otherwise, the two points would represent the same class. Furthermore, if two points on the same condensed zero divisor graph are adjacent to the same set of vertices and are also adjacent to one another, then at least one of the points must not annihilate itself, otherwise the two points would represent the same class. Their work was confined to vertices equal to 5. This has an implication that if the matrices were to be studied, it would not have a rank beyond 5.

A generalization of the zero divisor graphs associated to commutative rings was also provided by Afkhami *et al* in [1]. By means of matrix theory, the authors associated a simple graph to a commutative ring R which is called a generalized zero divisor graph of R denoted by Γ_R^n with $R^n \backslash \{0\}$ as the vertex set and two distinct vertices X and Y being adjacent if and only if there exists an $n \times n$ lower triangular matrix A over R whose entries on the main diagonal are nonzero and one of the entries on the main diagonal is regular such that $X^T AY = 0$ or $Y^T AY = 0$. They made a generalization for the cases when $\mid R \mid = p$ for $p = 2$ and $p > 2,\ n \geq 5$ in finding the graph theoretic properties like girth, diameter, clique number and chromatic number. Furthermore, it was concluded that if $\mid R \mid = p$ then $\Gamma_R^n \cong K_{p^n - 2}$. These results did not take into account cases of completely primary finite rings with unique maximal ideal $J(R)$.

Walwenda *et al* in [64] investigated the zero divisor graphs of finite rings in which the product of any two zero divisors lies in the coefficient subring for $Char(R) = p^k,\ k \geq 2$. That is, $(Z(R))^2 \subseteq GR(p^{kr}, p^k))$. The ring is completely primary finite satisfying $Z(R) = pR' \oplus U,\ (Z(R))^{k-1} = p^{k-1}R'$ and $(Z(R))^k = (0)$. Furthermore, they studied the geometric properties such as the diameter, the girth and the binding number of the graph $\Gamma(R)$. The research did not consider a study on the matrices of the classes of rings mentioned.

In 2016, Walwenda and Owino [65] did a research on the zero divisor graphs of a class of commutative completely primary finite rings. They considered R being completely primary finite with a unique maximal ideal $Z(R)$ satisfying $(Z(R))^{n-1} \neq (0),\ (Z(R))^n = (0)$ and classified the structures of their zero divisors by determining the geometrical and structural properties of $\Gamma(R)$ for every $Char(R) = p^k,\ k = 1, 2$ and $k \geq 3$. Our research has extended the classification of these rings to matrices and indices of the graphs .

Recently in 2020, Lao *et al* [31] classified the automorphisms of zero divisor graphs of Galois ring $R_\circ$ with $Z(R_\circ)$ being the Jacobson radical of $R_\circ$ and $R_\circ / Z(R_\circ)$ the Galois field of order p^r where p is prime and r positive integer. One of the major results obtained is stated in the following proposition.

Proposition 2.3.1. *(see Proposition 7 in [31]) Let* $Char(R) = p^2$ *and* $R_\circ = GR(p^{2r}, p^2)$ *and* $S_\circ = GR(p^2, p^2)$ *then*

$$| Aut(\Gamma(R_\circ)) |=| S_{p^r-1} |= \frac{1}{(p-2)!} | Aut(\Gamma(S_\circ)) | \sum_{i-1}^{r} p^{r-i}(p^r-2)!$$

and for $char(R_\circ) = p, Aut(\Gamma(R_\circ)) = \phi$, *where* ϕ *is an empty set.*

From this result, it was determined that the automorphism of $\Gamma(R_\circ)$ is an empty set if characteristic of the Galois ring is p. It was also clear from other findings in their research that if $R_\circ = GR(9, 9)$, then the cartesian product $\Gamma(R_\circ) \times \Gamma(R_\circ) \times \Gamma(R_\circ)$ is a hypercube of 3 dimensional space whose automorphism is isomorphic to $S_4 \times C_2$. Together with studies from other researchers, this work becomes handy in extending the classification of completely primary finite rings to their matrices.

2.4 Graph Indices

Some studies have also been performed on the graph indices such as the Wiener index and its invariants like the average disorder number and the average distance index. The Wiener index is a graph invariant that belongs to the molecules structure-descriptors called topological indices which are used for the design of molecules with desired properties [55].

Nathann *et al* in [41] conducted a research on Wiener index $W(G)$ and their line graphs $L(G)$. They demonstrated that if G is of a minimum degree at least two, then $W(G) \leq W(L(G))$. It was further illustrated that for every non-negative integer $g_\circ$, there exists $g > g_\circ$ such that there are infinitely many graphs G of girth g satisfying $W(G) = W(L(G))$. All the graphs considered were simple and undirected. Further, they showed that for integers $k, p, q \geq 1$, if $G = \Phi(k, p, q)$ with girth $g = 2k + 1$ then $W(L(G)) - W(G) = \frac{1}{2}(g^2 + (p - q)^2 + 5(p + q - 3) - 2g(p + q - 3))$ and for every non-negative integer h, there exist infinitely many graphs G of girth $g = h^2 + h + 9$ with $W(L(G)) = W(G)$(see Theorems 7 and 8 in [41]).

Mehdi *et al* in [36] conducted a study on the Wiener index of some graph operations. The Wiener indices constructed were from operations such as Mycielski's, generalized hierarchial product and $t^{th}-$ subdivision of graphs. The graphs studied were cycle graphs C_8, C_n and $k-$chromatic triangle-free graph G, where n is the order of the cycles. Further, it was noted that if G and H are graphs with $U \subset V(G)$, then the Wiener index of graph $\Gamma = G(U) \sqcap H$ is $W(\Gamma) =| V(H) | W(G)+ | V(G) |^2 W(H) + | V(H) | (| V(H) | -1)W(G(U))$.

Anwar *et al* in [4] conducted a research on entire Wiener index of graphs computed as $W^{\varepsilon}(G) = \Sigma_{\{x,y\} \in B(G)} d(x, y)$. The cases considered were for star graph S_n where $W^{\varepsilon}(S_n) = \frac{1}{2}(n - 1)(n - 8)$. For a complete graph K_n, $W^{\varepsilon}(K_n) = \frac{1}{4}n^2(n - 1)^2$ and finally, for any cycle C_n with $n \geq 3$, $W^{\varepsilon}(C_n) = \frac{1}{2}n^2(n - 1)$. They observed other cases with complete bipartite graph $K_{a,b}$ with $a, b \geq 2$ such as $W^{\varepsilon}(K_{a,b}) = (a + b)(a + b + \frac{1}{2}ab - 1) + ab(ab - 3)$. The cases were from graphs of rings of integers modulo n

Another index, the Zagreb index was introduced in [25] and given a further elaboration in [26]. Fundamental properties of the indices were given a summary in [43]. We obtain the bounds on $Z_1(\Gamma(R))$ and $Z_2(\Gamma(R))$ of the 3-radical zero and 4-radical zero completely primary finite rings in terms of n and m orders of $\Gamma(R)$ of maximum degree, $\Delta(\Gamma(R))$ and minimum degree $\delta((\Gamma(R)))$.

2.5 Matrices Associated with the Zero Divisor Graph $\Gamma(R)$

In this section, we review various studies done on the matrices associated with graphs, more specifically, the zero divisor graphs of finite rings.

2.5.1 Adjacency Matrices Associated with the Graph $\Gamma(R)$

Here, we look at some related studies done on the adjacency matrices of $\Gamma(R)$ of different classes of rings with a major focus on the recent studies done on the adjacency matrices of zero divisor graphs of completely primary finite rings.

A research on the adjacency matrix for zero-divisor graphs over finite rings of Gaussian integers was done by Pranjali *et al* in [54]. Their study concentrated on the characterization of such matrices corresponding to the Gaussian integer modulo n. The research calculated the number of zero divisors in each case and examined the orientation of the matrices and further generalized the order of the matrix in each case. The focus was on the rings $\mathbb{Z}_p[i]$, $\mathbb{Z}_{p^n}[i]$ and $\mathbb{Z}_{pq}[i]$. For the case $\mathbb{Z}_p[i]$, the specific attention was accorded to $p \equiv 2(mod\ 4),\ p \equiv 1(mod\ 4)$. For $\mathbb{Z}_{p^n},\ n > 1$ the rings in question were $p \equiv 1(mod\ 4), p \equiv 3(mod\ 4)$ and $p \equiv 2(mod\ 4)$. The last case was for the ring $\mathbb{Z}_n[i]$ when $n = p.q,\ n \equiv 2(mod\ 4)$. Among other findings, major result for $\mathbb{Z}_p[i],\ p \equiv 1(mod\ 4)$ is given in Theorem 2.5.1.

Theorem 2.5.1. *(See Theorem 2.1 in [54]) Let $\mathbb{Z}_p[i]$ be the ring of Gaussian integer modulo n. Consider $\mathbb{Z}_p[i],\ p \equiv 1(mod\ 4), p$ be prime. Let M be the adjacency matrix corresponding to the zero divisor graph of $\mathbb{Z}_p[i]$. The order of M in such case is always $2(p-1) \times 2(p-1)$.*

The study on these matrices was centered around the rings of Gaussian integer modulo n. Our study provides an advancement of this research by addressing the adjacency, Laplacian and Distance matrices of zero divisor graphs of completely primary finite rings with unique maximal ideal $J(R)$.

Pranjali *et al* in [53] in a separate work did an analysis of the adjacency matrix and the neighbourhood $N_G(v)$ associated with the zero divisor graph of finite commutative rings of direct product such that $R = R_1 \times R_2$. For p, prime, they investigated the matrix properties for $\mathbb{Z}_p \times \mathbb{Z}_p,\ p = 2, 3, 5$ and concluded that if $p \neq 2$, the adjacency matrix is always singular and the number of zero divisors is $2(p-1)$ resulting in a matrix of order $2(p-1) \times 2(p-1)$. Furthermore, the eigenvalues are $p-1$ for any $\mathbb{Z}_p \times \mathbb{Z}_p$. A further study was also on rings of type $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ for $p = 2, 3$ and concluded that the trace of the adjacency matrix corresponding to such rings is always a natural number, $n > 1$ and determinant 0 with the rank being $p-1$. The following theorem further explained the other result obtained.

Theorem 2.5.2. *(See Theorem b.2.3 in [53]) Let R be a finite commutative ring of cartesian product. Let G denote the zero divisor graph of the ring. Let $n(G)$ and $n(\bar{G})$ be the neighbourhood number of zero divisor graph of ring R and its compliment graph respectively then $n(G) = n(\bar{G})$.*

Their research was only confined to the adjacency matrix associated with the neighbourhood of every vertex $v_i \in \Gamma(R)$.

Patra and Baruah in [49] did an analysis of the adjacency matrix and the neighbourhood associated with the zero divisor graph for direct product of finite commutative rings of the form $\mathbb{Z}_p \times \mathbb{Z}_{p^2-2}$ and $\mathbb{Z}_p \times \mathbb{Z}_{2p}$ together with $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ where p is a prime integer. They established that the determinant of the adjacency matrices corresponding to the zero divisor graphs is 0, the matrices have a rank of 2 and are both symmetric and singular. Moreover, for $R = \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ the number of vertices for $G = \Gamma(\mathbb{Z}_{p^2})$ is $2p^2 - p - 1$, $\Delta(G) = p^2 - 1$ and $\delta(G) = p - 1$. For $\mathbb{Z}_p \times \mathbb{Z}_{2p}$ (where p is an odd prime number), then the number of vertices of $G = \Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ is $p^2 + 2p - 2,\ \Delta(G) = p^2 - 1$

and $\delta(G) = 1$. Finally, for $\mathbb{Z}_p \times \mathbb{Z}_{p^2-2}$ where p is an odd prime number and $p^2 - 2$ is prime, the zero divisor graph has $p^2 - p + 4$ vertices with $\Delta(G) = p^2 - 3$ and $\delta(G) = p - 1$. These results are summarized in the Theorem 2.5.3.

Theorem 2.5.3. *(See Theorem 2.4 in [49]) Let R_1 be a finite commutative ring such that $R_1 = \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ (p is a prime number). Let $G = \Gamma(R_1)$ be the zero-divisor graph with the vertex set $V = Z(R)^*$. Then $n_G(V) = 2\Delta(G) - \delta(G)$, where $n_G(V)$ is the neighbourhood number, $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G respectively.*

Another study was done by Rotich *et al* in [58] on the adjacency matrices of the Anderson-Livingston zero divisor graphs of Galois rings $R_\circ = GR(p^{kr}, p^k)$ whose set of non zero divisors is $pR_\circ - \{0\}$, with the order $\mid pR_\circ \mid = p^{(k-1)r} - 1$ resulting to the adjacency matrix of order $(p^{(k-1)r} - 1) \times (p^{(k-1)r} - 1)$. The authors presented their results for each characteristic of $R_\circ$. It was evident from their results that if $R_\circ = GR(p^r, p) \cong \mathbb{F}_{p^r}$, then the set of vertices $V(\Gamma(R_\circ)) = \emptyset$ and the adjacency matrix does not exist. Furthermore, for $R_\circ = GR(p^{2r}, p^2)$, the order of $[A]$ is $(p^r - 1) \times (p^r - 1)$ and the eigenvalues are $p^r - 2$ of multiplicity 1 and -1 of multiplicity $p^r - 2$. For general characteristic $p^k, k \geq 3$ and order p^{kr}, it was established that the adjacency matrix of the zero divisor graph of $R_\circ$ has a determinant of 0 and $0 \in \sigma_{point}([A])$, point spectrum of $[A]$. This left a gap on general characterization of the matrices of $\Gamma(R)$ where R is a finite ring.

Our current study is an extension of the research performed in [42] on the adjacency and incidence matrices of $\Gamma(R)$ of square radical zero where $Char R = p$ and p^2. The work detailed the graph properties of the ring including but not limited to completeness, girth, binding number, chromatic number and the maximum degree $\Delta(\Gamma(R))$. Moreover, a more specific study was done on their adjacency and incidence matrices and generalized the matrix algebraic properties with a particular inclusion of their quadratic forms definiteness. The following theorem was fundamental in [42] .

Theorem 2.5.4. *(See Theorem 10 in [42]) Let R be the square radical zero finite ring and $[A]$ and $[D]$ be the adjacency and incidence matrices respectively for $\Gamma(R)$ for positive integers h and r then:*

(i) $[A]$ and $[D]$ are both symmetric.

(ii) order of $[A] =$ order $[D]$

$$= \begin{cases} (p^{rh}-1) \times (p^{rh}-1), & \text{for char(R)=p} \\ (p^{(h+1)r}-1) \times (p^{(h+1)r}-1), & \text{for char(R)=}p^2. \end{cases}$$

(iii) rank$([A]) =$ trace$([A])$

$$= \begin{cases} p^{rh}-1, & \text{for char(R)=p} \\ p^{(h+1)r}-1, & \text{for char(R)=}p^2. \end{cases}$$

(iv) rank$([D]) = 1$ when char(R)=p or char(R)=p^2.

(v) The eigenvalues of $[A]$

$$= \begin{cases} p^{rh}-2 \text{ or } -1 \text{ of multiplicity } p^{rh}-2, & \text{for char(R)=p} \\ p^{(h+1)r}-2 \text{ or } -1 \text{ of multiplicity } p^{(h+1)r}-2, & \text{for char(R)=}p^2. \end{cases}$$

(vi) The eigenvalues of $[D]$

$$= \begin{cases} p^{rh}-1 \text{ or } 0 \text{ of multiplicity } p^{rh}-2, & \text{for char(R)=p} \\ p^{(h+1)r}-1 \text{ or } 0 \text{ of multiplicity } p^{(h+1)r}-2, & \text{for char(R)=}p^2. \end{cases}$$

Our extension is on Adjacency, Laplacian and Distance matrices of classes of 3-radical zero and 4-radical zero completely primary finite rings.

Katja *et al* in [29] performed a research on the eigenvalues of zero-divisor graphs of finite commutative rings. They provided computations for the nullity of $\Gamma(R)$ like the multiplicity of eigenvalues of $\Gamma(R)$. Moreover, the spectra of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ were precisely determined for a prime integer p. Furthermore, they introduced the graph product $\times_\Gamma$ with the property that $\Gamma(R) \cong \Gamma(R_1) \times_\Gamma \cdots \times_\Gamma \Gamma(R_r)$

whenever $R \cong R_1 \times \cdots \times R_r$. With this product, they found the relation between the number of vertices of the zero-divisor graph $\Gamma(R)$, the compressed zero-divisor graph, the structure of the ring R and the eigenvalues of $\Gamma(R)$.

2.5.2 Incidence Matrices Associated with the Graph $\Gamma(R)$

Namalai and Durairajan in [40] examined the linear codes with respect to the Hamming metric from the incidence matrices of zero-divisor graphs with vertex set of all non-zero zero-divisors of the ring $\mathbb{Z}_n$. They obtained the main parameters of the codes. For instance, it was noticed that if p_1 and p_2 are distinct primes then the main parameters of the $p-ary$ code generated by the incidence matrix of the graph $\Gamma(\mathbb{Z}_{p_1 p_2})$ is $[(p_1-1)(p_2-1),\ p_1+p_2-3,\ min(p_1-1,\ p_2-1)]$ for any p. Their findings however did not extend to other classes of rings.

Fulkerson and Gross in [23] performed a study on incidence matrices with the consecutive $1's$ property and noticed that a $(0,1)$-matrix A has the stated property for columns if there is a permutation matrix P such that the $1's$ in each column of PA occur consecutively. Further, they found that if A and B are $(0,1)$-matrices satisfying $A^T A = B^T B$, then either both A and B have the consecutive $1's$ property or neither does. Moreover, if A and B have the same number of rows and A has the property, then there is a permutation P such that $B = PA$.

Godsil, Sin and Xiang in [24] did a research on invariants of incidence matrices in which they focused on incidence of subsets of a finite set X where the consideration was on various relations between X_r and X_s where X_r denotes the set of subsets of size r, such as inclusion, empty intersection or, more generally, the intersection of fixed size t. The study done on incidence matrices did not shed some light on the incidence matrices of the zero divisor graphs of finite rings of higher characteristics.

Further study was done by Hamada in [27] on the rank of the incidence matrix where he found that if G is a graph and $A(G)$ is its incidence matrix and a row in $A(G)$ is a vector over $GF(2)$ in the vector space of graph G then the vectors $A_1, ..., A_n$ are linearly independent. Therefore, $rank(A) < n$. From the result, the authors conclude that if $A(G)$ is an incidence matrix of a connected graph G with n vertices, then the rank of $A(G)$ is $n-1$. It is important to note that their research was not tied to rings which are completely primary. Further, for our study, if the incidence matrices of these classes of rings would be studied, non square matrices would be obtained thus spectral properties would not be attained. This explains why this research is silent on the incidence matrices of the classes of rings.

2.5.3 Distance Matrices Associated with the Graph $\Gamma(R)$

The distance between two vertices v_i and v_j is denoted by $d(v_i, v_j)$ and it is defined as the length of the shortest path between vertices v_i and v_j [60]. The distance matrix of a graph G having n vertices is a symmetric matrix $[d_{ij}]$ whose entry d_{ij} is defined as $[d_{ij}] = d_{ij} = \begin{cases} d(v_i, v_j), & \text{if } i \neq j; \\ 0, & \text{if } i = j, 1 \leq i, j \leq n. \end{cases}$

From the existing literature, no work has been done on distance matrices of zero divisor graphs of completely primary finite rings. The available work surrounds a survey of the distance properties of some graphs of certain classes of rings, distance signless Laplacian spectra of some power graphs of integer modulo group, spectra of variants of distance matrices and digraphs among others. Distance matrices on graphs were introduced by Graham Pollack in 1971 to study a problem in communication. The authors of [28] compared and contrasted the techniques for distance Laplacian, distance signless Laplacian and the normalized distance Laplacian matrices. The digraphs were accorded a separate treatment and the similarities and differences between the graphs and digraphs were also discussed as they presented new results to compliment the existing ones such as on the unimodality of the characteristic polynomials, presentation of parameters by cospectrality for graphs and bounds on spectral radii of digraphs.

Bilal *et al* in [50] obtained the distance signless Laplacian spectrum of power graphs of integer modulo groups $\mathbb{Z}_n$. From their work, if n is a positive integer with prime factor decomposition $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ and if $\tau(n)$ denotes the number of positive divisors of n then, $\tau(n) = (n_1 + 1)(n_2 + 1) \cdots (n_r + 1)$.

Tamizh *et al* in [62] performed a survey on some of the distance aspects of the total graphs such as the diameter of the total graphs and their complements. Rings of integral domains and non integral domains were considered for study. For a commutative ring $R = \mathbb{Z}_2 \times \mathbb{Z}_4$, $diam(\Gamma(R)) \leq 3$ and $diam(\Gamma(R) = diam(\Gamma(T(R)))$. The properties regarding the graph complements were also investigated.

2.5.4 Laplacian Matrices Associated with the Graph $\Gamma(R)$

A lot of work has been done on the Laplacian matrices of the zero divisor graphs. Evidently, in Bilal *et al* [56] on the Laplacian eigenvalues of the zero-divisor graphs associated to the ring of integers modulo n. Their study was centered around finding the structure and the Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = p^{N_1} q^{N_2}$, where $p < q$ are primes and N_1, N_2 are positive integers. The following result was due to their research.

Theorem 2.5.5. *(See Theorem 5 in [56].) Let $\Gamma(\mathbb{Z}_n)$ be the zero divisor graph of order N, where $n = p^{N_1} q^{N_2}$ and $N_1 = 2m_1 \leq 2m_2 = N_2$. The Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consist of eigenvalues;*
$\{(p^i - 1)^{[\phi(p^{N_1-i} q^{N_2}) - 1]}, (q^j - 1)^{[\phi(p^{N_1} q^{N_2-j}) - 1]}, (pq^j - 1)^{[\phi(p^{N_1-l} q^{N_2-j}) - 1]}, \cdots,$
$(p^{m_1} q^k - 1)^{[\phi(p^{m_1} q^{N_2-k}) - 1]}, (p^{m_1} q^l - 1)^{[\phi(p^{m_1} q^{N_2-l}) - 1]}, \cdots, (p^{2m_1} q^k - 1)^{[\phi(q^{N_2-k}) - 1]}, (p^{2m_1} q^t - 1)^{[\phi(q^{N_2-t}) - 1]}\}$

where $i = 1, 2, \cdots, m_1, \cdots, N_1, j = 1, 2, \cdots, N_2, k = 1, 2, \cdots, m_2 - 1, l = m_2, \cdots, 2m_2$ *and* $t = m_2, \cdots, 2m_2 - 1$.

The research was limited to rings of integers modulo n. The research only focussed on rings of integers modulo n and did not extend beyond that.

Another research worth mentioning was done by Sripama *et al* in [61] concerning the Laplacian eigenvalues of the zero divisor graphs of the ring $\mathbb{Z}_n$. Their work was centered around the proof that $\Gamma(\mathbb{Z}_n)$ is a Laplacian integral for every prime p and positive integer $t \geq 2$. Further, they provided a proof that the Laplacian spectral radius and algebraic connectivity of $\Gamma(\mathbb{Z}_n)$ for most of the values of n are, respectively, the largest and the second smallest eigenvalues of the vertex weighted Laplacian matrix of a graph which is defined on the set of proper divisors of n.

A similar study was done by Pirzada *et al* in [51] describing the signless Laplacian spectrum of zero divisor graphs of the ring $\mathbb{Z}_n$. Their study was in an attempt to obtain the spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = p^z,\ z \geq 2$ in terms of signless Laplacian spectrum of its components and zeros of the characteristic polynomial of an auxilliary matrix. They obtained the following results.

Theorem 2.5.6. *(See Theorem 3.5 in [51].) Let* $n = p^z$ *where* $p > 2$ *is prime and* $z \geq 2$ *is a positive integer. Then the following hold:*

(i) If $z = 2$, *then the signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_n)$ *is* $\{2p - 4, (p - 3)^{[p-2]}\}$.

(ii) If $n = p^{2m}$ *for some positive integer* $m \geq 2$ *then the signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_n)$ *is* $\{(p-1)^{[\phi(p^{2m-1})-1]}, (p^2-1)^{[\phi(p^{2m-2})-1]}, \cdots, (p^{m-2}-1)^{[\phi(p^{m+2})-1]}, (p^{m-1}-1)^{[\phi(p^{m+1})-1]}\} \bigcup \{(p^m-3)^{[\phi(p^m)-1]}, (p^{m+1}-3)^{[\phi(p^{m-1})-1]}, (p^{2m-2}-3)^{[\phi(p^2)-1]}, (p^{2m-1}-3)^{[\phi(p)-1]}\}$.

(iii) If $n = p^{2m+1}$ *for some positive integer* $m \geq 2$, *then the signless Laplacian spectrum is*

$$\{(p-1)^{[\phi(p^{2m})-1]}, (p^2-1)^{[\phi(p^{2m}-1)-1]}, \cdots, (p^{m-1}-1)^{[\phi(p^{m+2})-1]}, (p^{m-1}-1)^{[\phi(p^{m+1})-1]}\} \bigcup$$
$$\{(p^{m+1}-3)^{[\phi(p^m)-1]}, (p^{m+2}-3)^{[\phi(p^m-1)-1]}, \cdots, (p^{2m-1}-3)^{[\phi(p^2)-1]}, (p^{2m}-3)^{[\phi(p)-1]}\}.$$

The selected literature shows that a lot of research done on Laplacian matrices have investigated the spectrum of $\Gamma(\mathbb{Z}_n)$. This leaves a significant gap to look at with regards to other rings. We therefore look at other algebraic properties and specifically of the Laplacian matrices of 3-radical zero and 4-radical zero completely primary finite rings.

From the aforementioned literature, it is worth remarking that the study of the zero divisor graphs of the finite rings and their geometrical properties has been a very active area of research. However, the information regarding the adjacency, Laplacian, distance matrices and indices of these zero divisor graphs is still not extensive in literature. In particular, no study has previously been done on the matrices and indices of the zero divisor graphs of classes of the 3-radical zero and 4-radical zero completely primary finite rings. We have made specific focus to the adjacency, Laplacian and the distance matrices of $\Gamma(R)$ of these classes of finite rings due to their similarities of algebraic properties and in how we have applied them in the interpretation of these graphs. Moreover, the adjacency and Laplacian matrices are connected through a matrix [D] whose diagonal is dominant of the degrees of vertices $v_i \in \Gamma(R)$ by the relationship $[L]_{ij} = [D]_{ij} - [A]_{ij}$. Further, we notice that the graphs in our classes of rings have their adjacency matrices similar to the distance matrices in entries except for the non connected vertices. This has motivated our research a great deal in finding an interpretation of zero divisor graphs of the classes of rings via their matrices and indices.

CHAPTER THREE

3-RADICAL ZERO COMPLETELY PRIMARY FINITE RINGS

3.1 Introduction

In this chapter, we have characterized the zero divisor graphs, some matrices associated with the zero divisor graph $\Gamma(R)$ of 3-radical zero finite rings. We begin by giving the construction for R given R' modules U and V whose generating sets contain s and t number of elements respectively such that for any $s,\ t = \frac{s(s+1)}{2}$. From the construction of R and the multiplication obtained, we have determined the structures of zero divisors $Z(R)$ and identified the adjacent and non-adjacent vertices hence the graph $\Gamma(R)$ is drawn and matrices constructed from it.

The following cases exist in literature on determination and characterization of unit groups of 3-radical zero completely primary finite rings [9, 10, 11, 12].

(i) $s = 2, t = 1, \lambda = 0, Char(R) = p, p^2, p^3$.

(ii) $s = 2, t = 1, \lambda \geq 1, Char(R) = p, p^2, p^3$.

(iii) $s = 2, t = 2, \lambda = 0, Char(R) = p,\ p^2,\ p^3$.

(iv) $s = 3, t = 1, \lambda \geq 1,\ Char(R) = p$.

3.2 Construction: Classes of the 3-Radical Zero Completely Primary Finite Rings

For any prime integer p and a positive integer r, let $R' = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and of characteristic p^k and consider the annihilator of $Z(R)$ be $(Z(R))^2$ so that $R = R' \oplus U \oplus V$ is an additive abelian group where U and V are finitely generated $R'-$modules. Let s and t be non negative numbers of elements in the generating sets $\{u_1, u_2, \cdots, u_s\}$ and $\{v_1, v_2, \cdots, v_t\}$ for U and V respectively.

Consider $t = \frac{s(s+1)}{2}$ for a fixed s and let $u_1, u_2, \cdots, u_s$ be commuting indeterminates over Galois ring $R' = GR(p^{kr}, p^k)$ where $1 \leq k \leq 3$. Then
$R = R' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$. The multiplication in R is given by the following relations:
$u_iu_j = u_ju_i = a_{ij}^k v,\ u_i^2 = a_{ii}^k v,\ u_i^3 = u_i^2u_j = u_iu_j^2 = 0$ for $1 \leq i, j \leq s$.
Here, (a_{ij}^k) defined in the multiplication of R is $t-$linearly independent matrices of dimension $s \times s$ with $1's$ in the $(i,j)^{th}$ and $(j,i)^{th}$ positions and $0-$elsewhere.
If $x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j$ and $y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j$ are any two elements in R where $x_\circ, y_\circ \in R',\ x_i, y_i, x_j, y_j \in R'/pR'$ then from the multiplication defined on R,

$$(x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j)(y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j) =$$

$$x_\circ y_\circ + \sum_{i=1}^{s}((x_\circ + pR')y_i + x_i(y_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(x_\circ y_j + x_j(y_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} a_{ij}^k x_i(y_j)^{\sigma_i})u_iu_j$$

where σ_i is an identity automorphism in R'. We show that the multiplication turns R into a commutative ring with identity $(1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0})$.
Let $x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j \in R$ with $x_\circ \in R'$ and $x_i, x_j \in R'/pR'$. We find $y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j \in R$ with $y_\circ \in R'$ and $y_i, y_j \in R'/pR'$ such that

$$(x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j)(y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j) =$$

$$(y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j)(x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j) = x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j.$$

If

$$x_\circ y_\circ + \sum_{i=1}^{s}((x_\circ + pR')y_i + x_i(y_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(x_\circ y_j + x_j(y_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} a_{ij}^k x_i(y_j)^{\sigma_i})u_iu_j$$

$$= x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j,$$

then $x_\circ y_\circ = x_\circ$,

$$\sum_{i=1}^{s}((x_\circ + pR')y_i + x_i(y_\circ + pR')^{\sigma_i})u_i = \sum_{i=1}^{s} x_iu_i$$

and

$$\sum_{i,j=1}^{s}(x_\circ y_j + x_j(y_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} a_{ij}^k x_i(y_j)^{\sigma_i})u_iu_j = \sum_{i,j=1}^{s} x_ju_iu_j.$$

So, $((x_\circ + pR')y_i)u_i = 0_R$ and $x_\circ = 1_{R'}$ for $i = 1, \cdots, s$. Since $u_i \neq 0$, $(x_\circ + pR')y_i = 0_{R'/pR'}$. But $x_\circ \in R'$ so, $y_i = 0_{R'/pR'}$. Finally, since $u_iu_j \neq 0$, $y_j = 0_{R'/pR'}$ thus $\sum_{i,j=1}^{s}(x_\circ y_j + x_j(y_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} a_{ij}^k x_i(y_j)^{\sigma_i}) = \bar{0}$. Therefore,

$$y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j = (1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0}).$$

In a similar way it can be demonstrated that

$$(y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j)(x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j) = x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j.$$

Next, we show that R is commutative if and only if σ_i for $i = 1, \cdots, s$ is an identity automorphism.

If σ_i is an identity automorphism then from the multiplication, R is commutative. In the converse, let R be commutative. We proceed to show that $\sigma_i = id_{R'}$. For every $x_\circ, y_\circ \in R'$ and $x_i, y_j \in R'/pR'$ we have that

$$(x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j)(y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j) =$$

$$x_\circ y_\circ + \sum_{i=1}^{s}((x_\circ + pR')y_i + x_i(y_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(x_\circ y_j + x_j(y_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} a_{ij}^k x_i(y_j)^{\sigma_i})u_iu_j.$$

$$= y_\circ x_\circ + \sum_{i=1}^{s}((y_\circ + pR')x_i + y_i(x_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(\sum_{i,j=1} a_{ij}^k x_i(y_j)^{\sigma_i} + x_\circ y_j + x_j(y_\circ)^{\sigma_i})u_iu_j.$$

Further, $x_i(y_\circ + pR')^{\sigma_i} - (y_\circ + pR')x_i = y_i(x_\circ + pR')^{\sigma_i} - (x_\circ + pR')y_i$. Since $x_i \neq y_i$ then x_i must be the identity in R' for the equality to be true.

R is commutative completely primary finite ring with identity $(1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0})$.

3.3 The 3-Radical Zero Completely Primary Finite Rings of Characteristic p

3.3.1 Construction I

Given a prime integer p and a positive integer r, let $R' = \mathbb{F} = GF(p^r)$ be a Galois field of order $q = p^r$. Suppose U and V are finitely generated $\mathbb{F}-$spaces with nonnegative number of elements s and t in the generating sets $\{u_s\}$ and $\{v_t\}$ respectively such that for $t = \frac{s(s+1)}{2}$ and s fixed, $R = \mathbb{F}_q \oplus \sum_{i=1}^{s} \mathbb{F}_q u_i \oplus \sum_{i,j=1}^{s} \mathbb{F}_q u_i u_j$ is an additive abelian group.

Consider the elements $a_\circ + \sum_{i=1}^{s} a_i u_i + \sum_{i,j=1}^{s} a_j u_i u_j$ and $b_\circ + \sum_{i=1}^{s} b_i u_i + \sum_{i,j=1}^{s} b_j u_i u_j \in R$, the multiplication in R is defined by

$$(a_\circ + \sum_{i=1}^{s} a_i u_i + \sum_{i,j=1}^{s} a_j u_i u_j)(b_\circ + \sum_{i=1}^{s} b_i u_i + \sum_{i,j=1}^{s} b_j u_i u_j) =$$

$$a_\circ b_\circ + \sum_{i=1}^{s}(a_\circ b_i^{\sigma_i} + a_i(b_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(a_\circ b_j + a_j(b_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} \alpha_{ij} a_i (b_j)^{\sigma_i})u_i u_j,$$

where σ_i is an $\mathbb{F}$ automorphism and (α_{ij}) is a t-linearly independent matrix of dimension s. If $\sigma_i = id_{\mathbb{F}}$ then R is turned into a commutative ring with identity $(1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0})$ by this multiplication. For the rings discussed in this section, we shall consider $\sigma_i = id_{\mathbb{F}}$.

Proposition 3.3.1. *Let R be the 3-radical zero finite ring of characteristic p in construction I and $Z(R)$ be the set of its zero divisors then:*

(i) $Z(R) = \sum_{i=1}^{s} \mathbb{F}_q u_i \oplus \sum_{i,j=1}^{s} \mathbb{F}_q u_i u_j$,

(ii) $(Z(R))^2 = \sum_{i,j=1}^{s} \mathbb{F}_q u_i u_j$ *and*

(iii) $(Z(R))^3 = (0)$.

Proof. (i) We need to show that every element $\mu \notin Z(R)$ is a unit.

Let $\mu = x_\circ + \sum_{i=1}^{s} x_i u_i + \sum_{i,j=1}^{s} x_j u_i u_j$ and $\mu' = y_\circ + \sum_{i=1}^{s} y_i u_i + \sum_{i,j=1}^{s} y_j u_i u_j$ be the multiplicative inverse of μ such that $\mu\mu' = 1_R$ then,

$$(x_\circ + \sum_{i=1}^{s} x_i u_i + \sum_{i,j=1}^{s} x_j u_i u_j)(y_\circ + \sum_{i=1}^{s} y_i u_i + \sum_{i,j=1}^{s} y_j u_i u_j) = (1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0}).$$

From the multiplication defined on R and $\sigma_i = id_{\mathbb{F}}$,

$$x_\circ y_\circ + \sum_{i=1}^{s}(x_\circ y_i + x_i y_\circ)u_i + \sum_{i,j=1}^{s}(x_\circ y_j + x_j y_\circ + \sum_{i,j=1}^{s} \alpha_{ij} x_i y_j)u_i u_j = (1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0}).$$

First,

$$x_\circ y_\circ = 1 \Longrightarrow y_\circ = x_\circ^{-1}.$$

For the second component,

$$\sum_{i=1}^{s}(x_\circ y_i + x_i y_\circ)u_i = 0 \Rightarrow \sum_{i=1}^{s} x_\circ y_i u_i + \sum_{i=1}^{s} x_i x_\circ^{-1} u_i = 0.$$

So,

$$\sum_{i=1}^{s} y_i u_i = -\sum_{i=1}^{s} x_i x_\circ^{-2} u_i.$$

Further, for the last summand,

$$\sum_{i,j=1}^{s}(x_\circ y_j + x_j y_\circ + \sum_{i,j=1}^{s} \alpha_{ij} x_i y_j)u_i u_j = \bar{0}.$$

$$\Rightarrow \sum_{i,j=1}^{s} x_\circ y_j u_i u_j + \sum_{i,j=1}^{s} x_j y_\circ u_i u_j + \sum_{i,j=1}^{s} \alpha_{ij} x_i y_j u_i u_j = \bar{0}.$$

Therefore,

$$\sum_{i,j=1}^{s} y_j u_i u_j = -\sum_{i,j=1}^{s}(x_j x_\circ^{-2} + \sum_{i,j=1}^{s} \alpha_{ij} x_i^2 x_\circ^{-2})u_i u_j.$$

This implies that $(x_\circ + \sum_{i=1}^{s} x_i u_i + \sum_{i,j=1}^{s} x_j u_i u_j)^{-1} =$

$$x_\circ^{-1} + (-\sum_{i=1}^{s} x_i x_\circ^{-2} u_i) + (-\sum_{i,j=1}^{s}(x_j x_\circ^{-2} + \sum_{i,j=1}^{s} \alpha_{ij} x_i^2 x_\circ^{-2})u_i u_j).$$

Therefore, $\mu \in R$ is invertible.

(ii) Since $Z(R) = \sum_{i=1}^{s} \mathbb{F}_q u_i \oplus \sum_{i,j=1}^{s} \mathbb{F}_q u_i u_j$ any element in $Z(R)$ can be represented as $0 + \sum_{i=1}^{s} x_i u_i + \sum_{i,j=1}^{s} x_j u_i u_j$. Let $0 + \sum_{i=1}^{s} x_i u_i + \sum_{i,j=1}^{s} x_j u_i u_j$ and $0 + \sum_{i=1}^{s} y_i u_i + \sum_{i,j=1}^{s} y_j u_i u_j$ be any two zero divisors in $Z(R)$. From the multiplication on R,

$$(0 + \sum_{i=1}^{s} x_i u_i + \sum_{i,j=1}^{s} x_j u_i u_j)(0 + \sum_{i=1}^{s} y_i u_i + \sum_{i,j=1}^{s} y_j u_i u_j) =$$

$$\sum_{i,j=1}^{s} \alpha_{ij} x_i y_j u_i u_j \in \sum_{i,j=1}^{s} \mathbb{F}_q u_i u_j.$$

(iii) Using the result in (ii) and the fact that $u_i^3 = u_i^2 u_j = u_i u_j^2 = 0$, the result is clear. □

3.3.2 Zero Divisor Graphs $\Gamma(R)$ and Matrices obtained from Classes of Rings in Construction I

Here we determine some graph geometric properties such as the order, completeness, diameter, degree, the girth and formulate the matrices from the graphs. We also investigate the matrix algebraic properties such as the rank, trace, determinant and order among others.

Proposition 3.3.2. *Consider R from Construction I and let p, s and r be positive integers. Then*

(i) $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+3s)}{2})r} - 1$,

(ii) $\Gamma(R)$ *is an incomplete graph,*

(iii) $diam((\Gamma(R)) = 2$,

(iv) Minimum degree, $\delta(\Gamma(R)) = p^{(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s)}{2}-1)r} - 1$, *and*

(v) $girth(\Gamma(R)) = 3$.

Proof. (i) Since $Char(R) = p$, $pu_i = pu_iu_j = 0$, and we have that $\mid \mathbb{F}_q u_i \mid = p^r$, $\mid \mathbb{F}_q u_i u_j \mid = p^r$, for any $i, j, = 1, 2, \cdots, s$ we obtain $\mid Z(R) \mid = p^{rs}.p^{\frac{sr(s+1)}{2}} = p^{(\frac{(s^2+3s)}{2})r}$. Therefore, $\mid Z(R)^* \mid = \mid Z(R) \backslash \{0\} \mid = p^{(\frac{(s^2+3s)}{2})r} - 1$. Since $\mid Z(R)^* \mid = \mid V(\Gamma(R)) \mid$, it follows that $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+3s)}{2})r} - 1$.

(ii) By part (ii) of Proposition 3.3.1, there are two elements of $Z(R)$ such that the product is non zero. This shows that not in all the vertices of $\Gamma(R)$ there is an edge and by the fact that $(Z(R))^2 \neq (0)$, $\Gamma(R)$ is incomplete.

(iii) From (ii), $\Gamma(R)$ is incomplete and with the fact that $Ann(Z(R)) = (Z(R))^2$, the result follows.

(iv) Let $V = \{v_1, v_2, \cdots, v_{p^{(\frac{(s^2+3s)}{2})r}-1}\}$ be the whole vertex set of $\Gamma(R)$. Let $K, S \subseteq V$ such that $K \subseteq ann(Z(R))^*$. This implies that $\mid K \mid = p^{(\frac{(s^2+3s)}{2}-1)r} - 1$. So,

$| S |= (p^{(\frac{(s^2+3s)}{2})r}-1)-(p^{(\frac{(s^2+3s)}{2}-1)r}-1) = p^{(\frac{(s^2+3s)}{2})r}-1-p^{(\frac{(s^2+3s)}{2}-1)r}+1$. Thus $| S |= p^{(\frac{(s^2+3s)}{2})r}-p^{(\frac{(s^2+3s)}{2}-1)r}$.

Therefore, $\delta(\Gamma(R)) = p^{(\frac{(s^2+3s)}{2})r}-p^{(\frac{(s^2+3s)}{2}-1)r}-1$ due to the minimal degree of the elements of S and for the avoidance of self loop for each vertex $v \in S$.

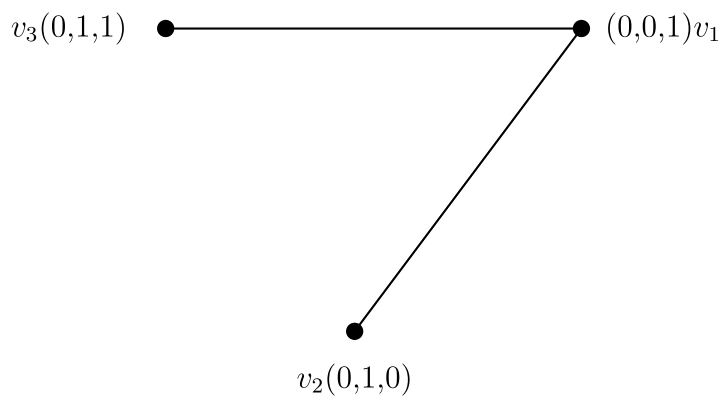
(v) Taking two vertices $v_1, v_2 \in Z(R)-(Z(R))^2$ where $v_1v_2 = 0$, each v_1, v_2 is adjacent to some $v_3 \in (Z(R))^2$. Thus $v_1 - v_2 - v_3 - v_1$ is the least polygon in $\Gamma(R)$.

□

Example 3.3.1. *For $p = 2$, fix $s = 1$, then $t = 1$ and $R = GF(2,2) \oplus GR(2,2) \oplus GR(2,2)$. So,*

$$Z(R)^* = \{(0,1,0), (0,1,1), (0,0,1)\}.$$

The zero divisor graph $\Gamma(R)$ is illustrated below.



We define the adjacency matrices as $[A] = [a_{ij}] = \begin{cases} 1, & v_iv_j{=}0; \\ 0, & \text{otherwise.} \end{cases}$ for any v_i, v_j. For an adjacency matrix $[A]_{ij}$, the diagonal matrix $[D]_{ij}$ whose diagonal entries are the degrees of vertices of $\Gamma(R)$, the Laplacian matrix

$$[L]_{ij} = [D]_{ij} - [A]_{ij}.$$

Next, we investigate the properties of the adjaceny, Laplacian and distance matrices associated with $\Gamma(R)$.

Proposition 3.3.3. *Let R be a ring given by Construction I and $\Gamma(R)$ its zero divisor graph. Then the adjacency matrix associated with $\Gamma(R)$ is of trace 0 with a spectral radius $p^r + 1$. Furthermore, the adjacency matrix, $[A]_{p^{(\frac{(s^2+3s)}{2})^r}-1}$ has the following properties.*

(i) $[A]_{p^{(\frac{(s^2+3s)}{2})^r}-1}$ *is symmetric.*

(ii) *rank* $([A]_{p^{(\frac{(s^2+3s)}{2})^r}-1}) = p^{(\frac{(s^2+3s)}{2})^r} - 2p^r$.

(iii) $Tr\ ([A]_{p^{(\frac{(s^2+3s)}{2})^r}-1}) = 0$, *and the spectral radius* $p^r + 1$.

(iv) $Det([A]_{p^{(\frac{(s^2+3s)}{2})^r}-1}) = 0$.

(v) *The eigenvalues* $\lambda\ ([A]_{p^{(\frac{(s^2+3s)}{2})^r}-1}) = \begin{cases} \pm\sqrt{2} \text{ and } 0. & \text{when } p=2; \\ 0, \quad \text{of multiplicity } 2p^r-1; & \\ -1, & \text{when } p \neq 2. \\ -p^r, \quad \text{and} & \\ p^r+1. & \end{cases}$

Proof. (i) Since every row vector

$(a_{11}, a_{12}, \cdots, a_{1(p^{(\frac{(s^2+3s)}{2})^r}-1)}), (a_{21}, a_{22}, \cdots, a_{2(p^{(\frac{(s^2+3s)}{2})^r}-1)}), \cdots,$
$(a_{(p^{(\frac{(s^2+3s)}{2})^r}-1)1}, a_{(p^{(\frac{(s^2+3s)}{2})^r}-1)2}, \cdots, a_{(p^{(\frac{(s^2+3s)}{2})^r}-1)(p^{(\frac{(s^2+3s)}{2})^r}-1)})$ of $[A]_{p^{(\frac{(s^2+3s)}{2})^r}-1}$ is a reflection of the corresponding element through the leading diagonal to every column

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{(p^{(\frac{(s^2+3s)}{2})^r}-1)1} \end{pmatrix} \cdots \begin{pmatrix} a_{1(p^{(\frac{(s^2+3s)}{2})^r}-1)} \\ a_{2(p^{(\frac{(s^2+3s)}{2})^r}-1)} \\ \vdots \\ a_{(p^{(\frac{(s^2+3s)}{2})^r}-1)(p^{(\frac{(s^2+3s)}{2})^r}-1)} \end{pmatrix},$$

it implies that $[A]_{p^{(\frac{(s^2+3s)}{2})^r}-1} = [A]^T_{p^{(\frac{(s^2+3s)}{2})^r}-1}$. Hence the symmetry of $[A]_{p^{(\frac{(s^2+3s)}{2})^r}-1}$.

(ii) Upon carrying out row operations on

$$[A]_{p^{(\frac{(s^2+3s)}{2})^r}-1} = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & 0 & 0 & \cdots 0_{p^{(\frac{(s^2+3s)}{2})^r}-2p^r} \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0_{p^{(\frac{(s^2+3s)}{2})^r}-1} & 0 & \cdots & \cdots & \cdots & 0_{p^{(\frac{(s^2+3s)}{2})^r}-1} \end{pmatrix},$$

we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & 0 & \cdots & & 0 \\ \vdots & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & 0 & \ddots & 1 & 0 & & \cdots 0_{p^{(\frac{(s^2+3s)}{2})^r - 2p^r}} \\ \vdots & \vdots & \vdots & 0 & \ddots & & \vdots \\ 0_{p^{(\frac{(s^2+3s)}{2})^r - 1}} & 0 & \cdots & \cdots & \cdots & & 0_{p^{(\frac{(s^2+3s)}{2})^r - 1}} \end{pmatrix}.$$

Let $V = \{v_1, v_2, v_3, \cdots, v_{p^{(\frac{(s^2+3s)}{2})^r - 2}}\}$ be the linearly independent set of vectors such that $v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, v_{p^{(\frac{(s^2+3s)}{2}-1)r}} =$ $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1_{p^{(\frac{(s^2+3s)}{2})^r - 2p^r}} \\ 0 \\ \vdots \\ 0_{p^{(\frac{(s^2+3s)}{2})^r - 1}} \end{pmatrix}$. Clearly, the set spans the matrix space implying that the $rank([A]_{p^{(\frac{(s^2+3s)}{2})^r - 1}}) = p^{(\frac{(s^2+3s)}{2})^r} - 2p^r$.

(iii) Given that $[A]_{p^{(\frac{(s^2+3s)}{2})^r - 1}}$ is the adjacency matrix for $\Gamma(R)$, it is justifiable that $Tr([A]_{p^{(\frac{(s^2+3s)}{2})^r - 1}}) = a_{11} + a_{22} + a_{33} + \cdots + a_{(p^{(\frac{(s^2+3s)}{2})^r - 1})(p^{(\frac{(s^2+3s)}{2})^r - 1})}$. Since $\Gamma(R)$ is a simple graph with no self loop for $v_i = 1, 2, \cdots, p^{(\frac{(s^2+3s)}{2})^r} - 1$, the leading diagonal entries are 0. Therefore, $\sum_{i=1}^{p^{(\frac{(s^2+3s)}{2})^r - 1}} a_{ii} = 0$. Further, since $Tr([A]_{p^{(\frac{(s^2+3s)}{2})^r - 1}}) = \sum_{j=1}^{p^{(\frac{(s^2+3s)}{2})^r - 1}} \lambda_j$. Therefore, the sum of eigenvalues $\lambda_j = -1 - p^r + p^r + 1 = 0$ and the spectral radius is $p^r + 1$.

(iv) Consider $[B]_{p^{(\frac{(s^2+3s)}{2})^r - 1}} = [A]_{p^{(\frac{(s^2+3s)}{2})^r - 1}} [C]^T_{p^{(\frac{(s^2+3s)}{2})^r - 1}}$ where $[C]_{p^{(\frac{(s^2+3s)}{2})^r - 1}}$ is the cofactor matrix of $[A]_{p^{(\frac{(s^2+3s)}{2})^r - 1}}$ then, $b_{ij} = \sum_k a_{ik} c_{jk}$ for c_{jk} is the jk minor of $[A]_{p^{(\frac{(s^2+3s)}{2})^r - 1}}$. If $i = j$, it corresponds to the determinant computation of $[A]_{p^{(\frac{(s^2+3s)}{2})^r - 1}}$ along the i^{th} row. Hence $b_{ii} = det([A]_{p^{(\frac{(s^2+3s)}{2})^r - 1}})$.

If $i \neq j$, this corresponds to the determinant computation of a matrix equal to $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ except that the row j has been overwritten by the contents of i^{th} row. But the determinant of a matrix with duplicated row is 0, hence $b_{ij} = 0. \Rightarrow$ $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}[C]^T_{p^{(\frac{(s^2+3s)}{2})r}-1} = det([A]_{p^{(\frac{(s^2+3s)}{2})r}-1})I$. If $det([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) \neq 0$ then we can write

$$A\frac{C^T}{det([A]_{p^{(\frac{(s^2+3s)}{2})r}-1})} = I \Leftrightarrow [A]^{-1}_{p^{(\frac{(s^2+3s)}{2})r}-1} = \frac{C^T}{det([A]_{p^{(\frac{(s^2+3s)}{2})r}-1})} = \frac{Adj[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}}{det([A]_{p^{(\frac{(s^2+3s)}{2})r}-1})}.$$

Therefore, $det([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) \neq 0 \Rightarrow [A]^{-1}_{p^{(\frac{(s^2+3s)}{2})r}-1}$ exists . Consequently, if $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ is singular then $det([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0$.

(v) For $p = 2$, we can obtain the eigenvalues by solving the characteristic equation $\mid \lambda I - A \mid = 0$. This means that $\lambda^3 - 2\lambda^2 = 0$ so, $\lambda(\lambda^2 - 2) = 0$. We have $\lambda = 0$ and $\lambda = \pm\sqrt{2}$.

For $p \neq 2$,

$\mid \lambda I - A \mid = 0$ results to a characteristic equation of the form $\lambda^{2p^r-1}(-(p^r+1)+\lambda)(1+\lambda)(p^r+\lambda) = 0$. Solving for λ in every factor results to $\lambda^{2p^r-1} = 0$. Thus $\lambda = 0$ of multiplicity $2p^r - 1$. For the second factor, $(-(p^r+1)+\lambda) = 0$ implies that $\lambda = p^r + 1$. For $(1+\lambda) = 0$, we have $\lambda = -1$ and lastly for $(p^r + \lambda) = 0$, we get $\lambda = -p^r$.

□

Proposition 3.3.4. *Consider* $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$*, the adjacency matrix associated with the zero divisor graph* $\Gamma(R)$ *where* R *is a ring of Construction I. Then for any fixed* $s \neq 1, r \in \mathbb{Z}^+$ *and* p*, prime integer, the following properties hold:*

(i) $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ *is symmetric.*

(ii) rank $([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = p^{(\frac{(s^2+3s)}{2}-1)r}$.

(iii) Tr $([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0$.

(iv) Det$([A]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0$.

(v) For p=2,

$$\lambda[A]_{p^{(\frac{(s^2+3s)}{2})r}-1} = \begin{cases} -1, & \text{of multiplicity } p^r; \\ 0, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2}-1)r}-1; \\ 1 \pm \sqrt{p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r} + 1}, & . \end{cases}$$

For $p \geq 3$,

The eigenvalues $\lambda[A]_{p^{(\frac{(s^2+3s)}{2})r}-1} =$

$$\begin{cases} -1, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2}-1)r}-2; \\ 0, & \text{of multiplicity } 2p^{(\frac{(s^2+3s)}{2}-1)r}-1; \\ \frac{p^{(\frac{(s^2+3s)}{2}-1)r}-2\pm\sqrt{9p^{(\frac{(s^2+3s)}{2}+1)r}-4p^{(\frac{(s^2+3s)}{2})r}-8p^{(\frac{(s^2+3s)}{2}-1)r}+4}}{2}, & . \end{cases}$$

Proof. The proof for (i) to (iv) have similar steps to the ones in Proposition 3.3.3.

(v) For $p = 2$, the equation $\mid [A]_{p^{(\frac{(s^2+3s)}{2})r}-1} - \lambda I_{p^{(\frac{(s^2+3s)}{2})r}-1} \mid= 0$ yields the characteristic equation for the adjacency matrix $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$. Let the eigenvalues of $[A]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ be $\lambda_1, \lambda_2, \cdots, \lambda_{p^{(\frac{(s^2+3s)}{2})r}-1}$. We can obtain the characteristic equation of the adjacency matrix as

$$-\lambda^{p^{(\frac{(s^2+3s)}{2}-1)r}-1}(1+\lambda)^{p^r}(\lambda^2 - 2\lambda - (p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r})) = 0.$$

Since $-\lambda^{p^{(\frac{(s^2+3s)}{2}-1)r}-1} = 0$ then $\lambda = 0$ of multiplicity $p^{(\frac{(s^2+3s)}{2}-1)r} - 1$ is an eigenvalue. Also, $(1+\lambda)^{p^r} = 0$ implies that $\lambda = -1$ of multiplicity p^r. By solving the quadratic equation

$\lambda^2 - 2\lambda - (p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r}) = 0$ in the last factor, we obtain

$$\lambda = \frac{2 \pm \sqrt{4 + 4p^{(\frac{(s^2+3s)}{2})r} + 4p^{(\frac{(s^2+3s)}{2}-1)r}}}{2} = \frac{2 \pm \sqrt{4(p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r} + 1)}}{2}$$

$$= \frac{2 \pm 2\sqrt{p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r} + 1}}{2} = 1 \pm \sqrt{p^{(\frac{(s^2+3s)}{2})r} + p^{(\frac{(s^2+3s)}{2}-1)r} + 1}.$$

For $p \neq 2$, we generally obtain the polynomial equation which when factorized results to $\lambda^{2p^{(\frac{(s^2+3s)}{2}-1)r}-1}(1+\lambda)^{p^{(\frac{(s^2+3s)}{2}-1)r}-2}((\lambda^2 - (p^{(\frac{(s^2+3s)}{2}-1)r} - 2)\lambda - (2p^{(\frac{(s^2+3s)}{2}+1)r} - p^{(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s)}{2}-1)r})) = 0$. Solving the equation gives the eigenvalues as $\lambda^{2p^{(\frac{(s^2+3s)}{2}-1)r}-1} = 0$ which implies that $\lambda = 0$ of multiplicity $2p^{(\frac{(s^2+3s)}{2}-1)r} - 1$. The second factor

$(1+\lambda)^{p^{(\frac{(s^2+3s)}{2}-1)r}-2} = 0$ yields $\lambda = -1$ of multiplicity $p^{(\frac{(s^2+3s)}{2}-1)r} - 2$. Finally, solving the quadratic part results to

$$\lambda = \frac{(p^{(\frac{(s^2+3s)}{2}-1)r} - 2) \pm \sqrt{(p^{(\frac{(s^2+3s)}{2}-1)r} - 2)^2 + 4(2p^{(\frac{(s^2+3s)}{2}+1)r} - p^{(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s)}{2}-1)r})}}{2}.$$

Expanding the discriminant yields the expression $p^{(\frac{(s^2+3s)}{2}+1)r} - 4p^{(\frac{(s^2+3s)}{2}-1)r} + 4 + 8p^{(\frac{(s^2+3s)}{2}+1)r} - 4p^{(\frac{(s^2+3s)}{2})r} - 4p^{(\frac{(s^2+3s)}{2}-1)r}$ so that

$$\lambda = \frac{(p^{(\frac{(s^2+3s)}{2}-1)r} - 2) \pm \sqrt{9p^{(\frac{(s^2+3s)}{2}+1)r} - 4p^{(\frac{(s^2+3s)}{2})r} - 8p^{(\frac{(s^2+3s)}{2}-1)r} + 4}}{2}.$$

□

Proposition 3.3.5. *Consider $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$, the Laplacian matrix associated with $\Gamma(R)$ of the ring in Construction I. Then for any positive integer r, prime integer p, the following properties hold:*

(i) $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ *is symmetric,*

(ii) $rank([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = p^{(\frac{(s^2+3s)}{2})r} - 2$,

(iii) $Tr([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \begin{cases} 4, & \text{when } s = 1, p = 2; \\ (p^r - 1)(2p^{(\frac{(s^2+3s)}{2})r} - p^r - 2), & \text{for any } s, p \geq 2. \end{cases}$

(iv) $Det([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0$, *and*

(v) The eigenvalues $\lambda[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ *are* $0, 1$ *and* 3 *when* $s = 1, p = 2$.
For any $s, p \geq 2$ *the eigenvalues* $\lambda[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$
$= \begin{cases} 0, & \\ p^{(\frac{(s^2+3s)}{2})r} - 1, & \text{of multiplicity } p^r - 1; \\ p^r - 1, & \text{of multiplicity } 2p^r - 1. \end{cases}$

Proof. (i) Can be drawn from the previous Proposition 3.3.4 since the steps are similar.

(ii) We conduct a row operation on $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ to obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & . & . & -1 \\ 0 & 1 & 0 & 0 & . & . & -1 \\ 0 & . & \ddots & 0 & . & . & -1 \\ \vdots & . & . & 1_{p^{(\frac{(s^2+3s)}{2})r}-2} & 0 & 1 & -1 \\ 0 & 0 & . & . & . & 0 & 0 \end{pmatrix}.$$

This results to $p^{(\frac{(s^2+3s)}{2})r}-2$ non-zero rows in $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$, hence its rank.

(iii) For $p=2$ and $s=1$, the $\Gamma(R)$ obtained has only 1 vertex of maximum degree 2 and 2 vertices of minimum degree 1. This leads to a Laplacian matrix of order 3×3 whose main diagonal entries are $2,1$ and 1 hence a trace of 4.

For any $s,p\geq 2$,
$\mid Z(R)^* \mid = p^{(\frac{(s^2+3s)}{2})r}-1$ and each $v_i \in Ann(Z(R)^*)$ has degree $p^{(\frac{(s^2+3s)}{2})r}-2$ and $\mid Ann(Z(R)^*) \mid = p^r-1$. Therefore any $v_j \notin Ann(Z(R)^*)$ is of degree p^r-1 because every such $v_j \notin Ann(Z(R)^*)$ is only adjacent to $v_i \in Ann(Z(R)^*)$.
Partitioning the vertices of $\Gamma(R)$ into disjoint subsets V_1 and V_2 such that
$V_1 = \{v_j | v_j \notin Ann(Z(R)^*)\}$ and $V_2 = \{v_i | v_i \in Ann(Z(R)^*)\}$,
$\mid V_2 \mid = p^r-1$ and $\mid V_1 \mid = p^{(\frac{(s^2+3s)}{2})r}-1-(p^r-1) = p^{(\frac{(s^2+3s)}{2})r}-p^r$. Since the trace, $Tr([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \sum_{i=1}^{p^{(\frac{(s^2+3s)}{2})r}-1} l_{ii}$, where every l_{ii} is an element of the diagonal matrix $[D]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ whose entries are degrees of vertices in $\Gamma(R)$, we have that $Tr([L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = (p^{(\frac{(s^2+3s)}{2})r}-2)(p^r-1)+(p^r-1)(p^{(\frac{(s^2+3s)}{2})r}-p^r)$. This results to $(p^r-1)(p^{(\frac{(s^2+3s)}{2})r}-2+p^{(\frac{(s^2+3s)}{2})r}+p^r) = (p^r-1)(2p^{(\frac{(s^2+3s)}{2})r}-p^r-2)$.
(iv) Steps in obtaining the singularity of $[L]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ are similar to the one in Proposition 3.3.3.
(v) For $p=2, s=1$, the eigenvalues for the 3×3 Laplacian matrix are easy to obtain. When $p\geq 2$ for any $s\geq 2$, the equation $\mid (\lambda I_{p^{(\frac{(s^2+3s)}{2})r}-1} - [L]_{p^{(\frac{(s^2+3s)}{2})r}-1}) \mid = 0$ gives the characteristic polynomial equation of the form $-\lambda((-(p^{(\frac{(s^2+3s)}{2})r}-1)+\lambda)^{p^r-1}(-(p^r-1)+\lambda)^{2p^r-1}) = 0$. On solving for λ in each factor, we obtain $-\lambda = 0$ which means that $\lambda = 0$. For the equation $(-(p^{(\frac{(s^2+3s)}{2})r}-1)+\lambda)^{p^r-1} = 0$, we have that $\lambda = (p^{(\frac{(s^2+3s)}{2})r}-1)$ of multiplicity p^r-1. Finally, $(-(p^r-1)+\lambda)^{2p^r-1} = 0$ implies that $\lambda = p^r-1$, of multiplicity $2p^r-1$. This establishes (v). □

Example 3.3.2. *Consider the graph in Example 3.3.1. The set of vertices for $\Gamma(R)$ is given by*

$$\{v_1, v_2, v_3\} = \{(0,0,1), (0,1,0), (0,1,1)\}$$

The row vectors for the adjacency matrix formed are $(a_{11}, a_{12}, a_{13}) = (0, 1, 1), (a_{21}, a_{22}, a_{23}) = (1, 0, 0), (a_{31}, a_{32}, a_{33}) = (1, 0, 0)$. *Thus the adjacency and Laplacian matrices associated with* $\Gamma(R)$ *in this example are;*

$$[A] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3\times 3}$$

and the Laplacian matrix for $\Gamma(R)$ *in this case is given by*

$$[L] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} - \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}_{3\times 3}.$$

Next, we take an analysis of the distance matrix of $\Gamma(R)$ of the ring in Construction I. Recall that the distance matrix of a graph G having n vertices is a symmetric matrix $[d_{ij}]$ whose entry d_{ij} is defined as $d_{ij} = \begin{cases} d(v_i, v_j), & \text{if } i \neq j; \\ 0, & \text{if } i = j,\ 1 \leq i, j \leq n. \end{cases}$
The following result describes the matrix algebraic properties of $[d_{ij}]$ of this class of rings.

Proposition 3.3.6. *Consider* $[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}$, *distance matrix associated with* $\Gamma(R)$ *of* R *in Construction I. For* $r \in \mathbb{Z}^+$, p *prime,*

(i) $Det([d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \begin{cases} 4, & \textit{when } p = 2, s = 1; \\ -(p^{(\frac{(s^2+3s)}{2})r} - 1)^2, & \textit{for any } s, p \geq 2. \end{cases}$

(ii) $Tr([d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0.$

(iii) $rank([d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \begin{cases} 3, & \textit{when } p = 2, s = 1; \\ p^{(\frac{(s^2+3s)}{2})r} - 1, & \textit{for any } s, p \geq 2, \end{cases}$ *and*

(iv) When $p = 2, s = 1$ *the eigenvalues* $\lambda[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ *are* $1 \pm \sqrt{3}$ *and* -2

For any $s, p \geq 2$, *the eigenvalues* $\lambda[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}$

$$= \begin{cases} -1, & \textit{multiplicity } p^{(\frac{(s^2+3s)}{2})r} + (p^r + 1); \\ 1 - p^r, & \\ \frac{1}{2}((p^{(\frac{(s^2+3s)}{2})r} + 2) \pm \sqrt{p^{(\frac{2(s^2+3s)}{2})r} + 4p^{(\frac{(s^2+3s)}{2})r} + 4p^r}) & . \end{cases}$$

Proof. (i) When $p = 2, s = 1$, we obtain the distance matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$ from example 3.3.1.

Expanding the matrix along the first row, we clearly obtain the determinant to be 4.

When $p \geq 2$ for any fixed s, we obtain the distance matrix to be of the form

$$\begin{pmatrix} 0 & 1 & \cdots & \cdots & \cdots & 1 & 1_{p^{(\frac{(s^2+3s)}{2})r}-1} \\ 1 & 0 & 1 & \cdots & 1 & \cdots & \cdots \\ \vdots & \cdots & \ddots & & & & 1_{p^{(\frac{(s^2+3s)}{2})r}-1} \\ 1 & 1 & 2 & 0 & 2 & \cdots & 2 \\ \vdots & \cdots & \vdots & \cdots & \ddots & & \vdots \\ 1 & 1 & \cdots & 2 & \cdots & & 0 \end{pmatrix}.$$

Expanding along the first row, we obtain the determinant to be

$$-(p^{(\frac{(s^2+3s)}{2})r} - 1)(p^{(\frac{(s^2+3s)}{2})r} - 1) = -(p^{(\frac{(s^2+3s)}{2})r} - 1)^2.$$

(ii) Due to the fact that $d(v_i, v_i) = 0$, this results to a distance matrix $[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ with $0'$s entirely in the main diagonal. Therefore,

$$Tr([d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}) = \sum_{i=1}^{p^{(\frac{(s^2+3s)}{2})r}-1} d(v_i, v_i) = 0.$$

(iii) When $p = 2, s = 1$, the rank of the matrix in (i) upon carrying out row reductions is obtained to be 3.

When $p \geq 2$ for any fixed s, we can obtain the rank of the distance matrix $[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ by conducting a row operation on it which reduces to the echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 2 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 2 \\ \vdots & & \ddots & & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & \cdots & 0 & 2 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & -1 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & & 0 & 1 & -1_{p^{(\frac{(s^2+3s)}{2})r}-2} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \cdots & 1_{p^{(\frac{(s^2+3s)}{2})r}-1} \end{pmatrix}.$$

From the reduced echelon form above, there are $p^{(\frac{(s^2+3s)}{2})r} - 1$ linearly independent vectors which span the row space of $[d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1}$ resulting to $p^{(\frac{(s^2+3s)}{2})r} - 1$ nonzero rows, hence its rank.

(iv) Solving the equation $| \lambda I - [d_{ij}]_{p^{(\frac{(s^2+3s)}{2})r}-1} |= 0$, results to the characteristic equation

$(1+\lambda)((p^r-1)+\lambda)^{p^{(\frac{(s^2+3s)}{2})r}+(p^r+1)}(\lambda^2-(p^{(\frac{(s^2+3s)}{2})r}+2)\lambda-(p^r-1))=0$. We obtain the eigenvalues by finding the solution for λ in every factor of the equation as follows: Clearly, $(\lambda+1)=0$ implies that $\lambda=-1$. Further, $((p^r-1)+\lambda)^{p^{(\frac{(s^2+3s)}{2})r}+(p^r+1)}=0$, we obtain $\lambda = 1-p^r$ of multiplicity $p^{(\frac{(s^2+3s)}{2})r}+(p^r+1)$.

Finally, for the quadratic part $(\lambda^2-(p^{(\frac{(s^2+3s)}{2})r}+2)\lambda-(p^r-1))=0$, we can obtain $\lambda = \frac{1}{2}((p^{(\frac{(s^2+3s)}{2})r}+2) \pm \sqrt{(p^{(\frac{(s^2+3s)}{2})r}+2)^2+4(p^r-1)})$

which on expansion yields

$$\lambda = \frac{1}{2}((p^{(\frac{(s^2+3s)}{2})r}+2) \pm \sqrt{p^{(\frac{2(s^2+3s)}{2})r}+4p^{(\frac{(s^2+3s)}{2})r}+4+4(p^r-1)})$$

and simplifies to $\lambda = \frac{1}{2}((p^{(\frac{(s^2+3s)}{2})r}+2) \pm \sqrt{p^{(\frac{2(s^2+3s)}{2})r}+4p^{(\frac{(s^2+3s)}{2})r}+4p^r})$. □

Example 3.3.3. *Let $s=1$, $p=3$. Then, $R = GR(3,3) \oplus GR(3,3) \oplus GR(3,3)$. The distance matrix in this case is given by*

$$[d_{ij}] = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 0 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 0 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 & 0 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}_{8\times 8}.$$

3.4 The 3-Radical Zero Completely Primary Finite Rings of Characteristic p^2

3.4.1 Construction II

Let $R' = GR(p^{2r}, p^2)$ be a Galois ring of order p^{2r} and of characteristic p^2. Let U and V be finitely generated R'-modules with $\{u_1, u_2, \cdots, u_s\}$ and $\{v_1, v_2, \cdots, v_t\}$ being the generating set such that the nonnegative integers s, t are the number of elements in the respective generating sets. Then for fixed s and $t = \frac{s(s+1)}{2}$,

$R = R' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$ is an additive abelian group. Define multiplication in R by

$$(a_\circ + \sum_{i=1}^{s} a_i u_i + \sum_{i,j=1}^{s} a_j u_i u_j)(b_\circ + \sum_{i=1}^{s} b_i u_i + \sum_{i,j=1}^{s} b_j u_i u_j) =$$

$$a_\circ b_\circ + \sum_{i=1}^{s}((a_\circ + pR')b_i + a_i(b_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(a_\circ b_j + a_j(b_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} \alpha_{ij}^k a_i(b_j)^{\sigma_i})u_i u_j.$$

R is thus turned by the multiplication into a commutative ring with identity $(1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0})$.

For the rings discussed in this section, we shall consider $\sigma_i = id_{R'}$.

Case I: when $pu_i = 0$

Proposition 3.4.1. *Consider R from Construction II and let $pu_i = 0$. Then zero divisors set $Z(R)$ satisfies the following properties:*

(i) $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_i u_j$,

(ii) $(Z(R))^2 = \sum_{i,j=1}^{s} R'u_i u_j$, *and*

(iii) $(Z(R))^3 = (0)$.

Proof. (i). Assume $b \in R'$ and that $b \notin pR'$ and $y \in Z(R)$. Then for some $r \in \mathbb{Z}^+$, binomial theorem gives $(b+y)^{p^r} = \binom{p^r}{0} b^{p^r} y^0 + \binom{p^r}{1} b^{p^r-1} y^1 + \cdots + \binom{p^r}{p^r} b^0 y^{p^r} = b^{p^r} + p^r b^{p^r-1} y + \cdots + y^{p^r} = b^{p^r} + y_1$ where $y_1 \in Z(R)$ and $p^r b^{p^r-1} y = p^{r+1} b^{p^r-2} y^2 = \cdots = 0, y^{p^r} = y_1$ but $b^{p^r} + y_1 = b + y_2$, where $y_2 \in Z(R)$. Consequently, $(b+y_2)^{p^r-1} = 1 + y_3$ for $y_3 \in Z(R)$ and $(1+y_3)^{p^2} = 1$. Therefore, $(((b+y)^{p^r})^{p^r-1})^{p^2}$ shows the invertibility of $b+y$.

Further, $\mid Z(R) \mid = p^{(\frac{(s^2+3s+2)}{2})r}$ and $\mid R'/pR' + Z(R) \mid = (p^{2r} - 1)p^{(\frac{(s^2+3s+2)}{2})r} = p^{(\frac{(s^2+3s+4)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}$ which gives the order of the units in R for which $(R'/pR')^* + Z(R) = R - Z(R)$ since $R'/pR' \cong R'$ and $\mid R' \mid = p^{2r}$. This proves that all elements not in $Z(R)$ are in R^*.

(ii) From the defined multiplication on R, consider $(pa_\circ + \sum_{i=1}^{s} a_i u_i + \sum_{i,j=1}^{s} a_j u_i u_j)$

and $(pb_\circ + \sum_{i=1}^{s} b_i u_i + \sum_{i,j=1}^{s} b_j u_i u_j) \in Z(R)$. Then the product

$$(pa_\circ + \sum_{i=1}^{s} a_i u_i + \sum_{i,j=1}^{s} a_j u_i u_j)(pb_\circ + \sum_{i=1}^{s} b_i u_i + \sum_{i,j=1}^{s} b_j u_i u_j) =$$

$$(\sum_{i,j=1}^{s} \alpha_{ij} a_i b_j) u_i u_j.$$

Adding the product finitely, we obtain $(Z(R))^2 \subseteq \sum_{i,j=1}^{s} R' u_i u_j$.

Conversely, let $x \in \sum_{i,j=1}^{s} R' u_i u_j$, then $x = yw$ and $w \in Z(R)$. $\exists\ u_i, u_j \in Z(R)$ such that $v = u_i u_j$. So, $y u_i u_j \in (Z(R))^2 \Rightarrow \sum_{i,j=1}^{s} R' u_i u_j \in (Z(R))^2$.

Therefore, $(Z(R))^2 = \sum_{i,j=1}^{s} R' u_i u_j$.

(iii). The product $Z(R)(Z(R))^2 = (Z(R))^2 Z(R) = (0)$ since

$$(Z(R))^2 \subseteq Ann(Z(R)) = \{pa_\circ + \sum_{i=1}^{s} a_i u_i + \sum_{i,j=1}^{s} a_j u_i u_j | a_i + a_j \equiv 0 (mod p)\}.$$

Since $RZ(R) = Z(R)R = Z(R)$, the set $Z(R)$ is an ideal. Its uniqueness and maximality follows from the fact that any other ideal, distinct from $Z(R)$ contains a unit and is therefore the whole ring R. □

3.4.2 The Graphs $\Gamma(R)$ and Matrices obtained from Classes of Rings in Construction II

Proposition 3.4.2. *Consider R from Construction II. Then for p prime, $r \in \mathbb{Z}^+$ and $pu_i = 0$, $\Gamma(R)$ has the following properties:*

(i) $| V(\Gamma(R)) | = p^{(\frac{(s^2+3s+2)}{2})r} - 1$,

(ii) $\Gamma(R)$ *is an incomplete graph,*

(iii) $diam((\Gamma(R)) = 2$,

(iv) minimum degree, $\delta(\Gamma(R)) = p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r} - 1$, *and*

(v) $girth(\Gamma(R)) = 3$.

Proof. (i) Since $char(R) = p^2$, $pu_i = 0$, and we have that $| R' u_i | = p^r, | R' u_i u_j | = p^r, | pR' | = p^r$ which is true for every $i, j = 1, \cdots s$. We obtain $| Z(R) | = p^{(\frac{(s^2+3s+2)}{2})r}$.

Therefore, $\mid Z(R)\backslash\{0\} \mid= p^{(\frac{(s^2+3s+2)}{2})r} - 1$ since $\mid Z(R)^* \mid=\mid V(\Gamma(R)) \mid\Longrightarrow\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+3s+2)}{2})r} - 1$.

(ii) Follows from the fact that $(Z(R))^2 \neq (0)$.

(iii) From the incompleteness of $\Gamma(R)$ in (ii), $Ann(Z(R)) = (Z(R))^2$, there exist some two non adjacent vertices $x, y \in V(\Gamma(R))$ so that for some $z \in Ann(Z(R))$, the supremum distance $d\{x, y\} = 2$ hence the diameter.

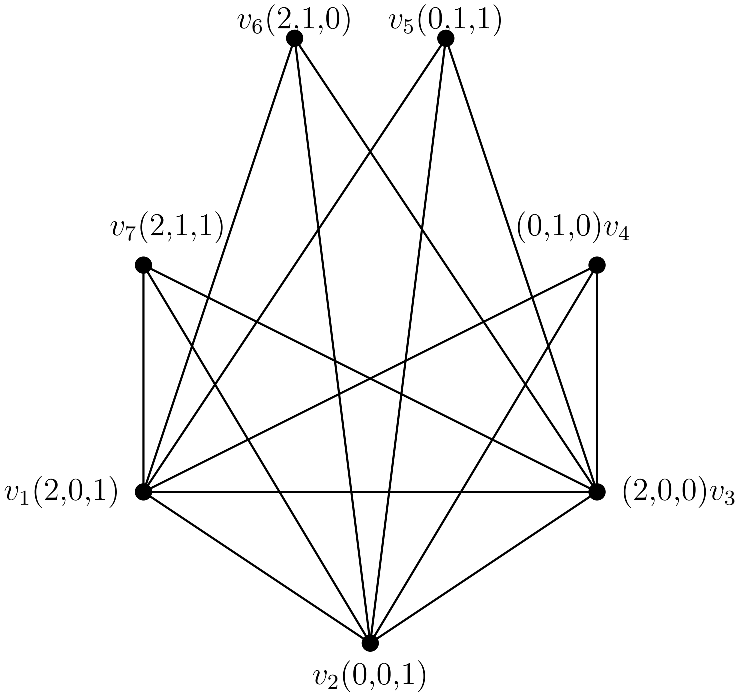
(iv) Let $V = \{v_1, v_2, \cdots, v_{p^{(\frac{(s^2+3s+2)}{2})r}-1}\}$ be the vertex set for $\Gamma(R)$. Let $K, S \subseteq V$ such that $K \subseteq ann(Z(R))^*$. This implies $\mid K \mid= p^{(\frac{(s^2+3s)}{2})r} - 1$ and $\mid S \mid= (p^{(\frac{(s^2+3s+2)}{2})r} - 1) - (p^{(\frac{(s^2+3s)}{2})r} - 1) = p^{(\frac{(s^2+3s+2)}{2})r} - 1 - p^{(\frac{(s^2+3s)}{2})r} + 1$ which simplifies to $p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}$. Therefore, $\delta(\Gamma(R)) = (p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}) - 1 = p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r} - 1$ due to the minimal degree of the elements of S and for the avoidance of self loop for each vertex $v \in S$. Hence minimum degree $\delta(\Gamma(R))$.

(v) The Proof follows a similar pattern as part (v) of Proposition 3.3.2. □

Example 3.4.1. *Let $s = 1$ and $p = 2$. Then $R = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

$$Z(R)^* = \{(2, 1, 1), (0, 0, 1), (2, 0, 0), (0, 1, 0), (0, 1, 1), (2, 1, 0), (2, 0, 1)\}.$$

From the multiplication on R, we obtain $\Gamma(R)$ as shown below.



Proposition 3.4.3. *Consider* $[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$*, the adjacency matrix associated with* $\Gamma(R)$ *for* R *in Construction II. Then for* $pu_i = 0$ *and any fixed* $s, r \in \mathbb{Z}^+$ *and* p *prime,*

(i) $[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ *is symmetric,*

(ii) $rank([A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = p^{(\frac{(s^2+3s)}{2})r} - 1,$

(iii) $Tr([A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = 0,$

(iv) $Det([A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = 0,$ *and*

(v) For $p = 2$*, the eigenvalues* $\lambda[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$

$$= \begin{cases} -1, & \text{of multiplicity } p^r; \\ 0, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2})r} - 1; \\ 1 \pm \sqrt{p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} + 1}, & . \end{cases}$$

For $p \neq 2$*, eigenvalues* $\lambda[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$

$$= \begin{cases} -1, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2})r} - 2; \\ 0, & \text{of multiplicity } 2p^{(\frac{(s^2+3s)}{2})r} - 1; \\ \frac{(p^{(\frac{(s^2+3s)}{2})r}-2) \pm \sqrt{p^{2(\frac{(s^2+3s)}{2})r} - 8p^{(\frac{(s^2+3s)}{2})r} + 4p^{(\frac{(s^2+3s+2)}{2})r} + 4}}{2}, & . \end{cases}$$

Proof. Proofs for (i), (iii) and (iv) follow similar fashion as proof of Proposition 3.3.3. We provide proofs for (ii) and (v) as follows:

(ii) Upon carrying out a row operation on

$$[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1} = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & & 1 \\ 1 & 0 & 1 & 1 & \cdots & & 1 \\ \vdots & \vdots & 0 & 1 & \cdots & & \vdots \\ 1 & 1 & 1 & 0 & 0 & & \cdots 0_{p^{(\frac{(s^2+3s)}{2})r}} \\ \vdots & \vdots & \vdots & 0 & \cdots & & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & & 0 \end{pmatrix},$$

we obtain its reduced echelon form matrix as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & 0 & \cdots & & 0 \\ \vdots & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & 0 & 0 & 1 & 0 & & \cdots 0_{p^{(\frac{(s^2+3s)}{2})r}} \\ \vdots & \vdots & \vdots & 0 & \cdots & & \vdots \\ 0_{p^{(\frac{(s^2+3s+2)}{2})r}-2} & 0 & \cdots & \cdots & \cdots & & 0_{p^{(\frac{(s^2+3s+2)}{2})r}-1} \end{pmatrix}.$$

Let $\{v_1, v_2, v_3, \cdots, v_{p^{(\frac{(s^2+3s)}{2})r}}\}$ be the linearly independent set of vectors such that $v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, v_{p^{(\frac{(s^2+3s)}{2})r}} = \begin{pmatrix} 0 \\ \vdots \\ 1_{p^{(\frac{(s^2+3s)}{2})r}-1} \\ 0 \\ \vdots \\ 0_{p^{(\frac{(s^2+3s+2)}{2})r}-1} \end{pmatrix}$. Clearly, the set spans the whole of the matrix space. Therefore the $rank([A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = p^{(\frac{(s^2+3s)}{2})r} - 1$.

(v) For $p = 2$, $\mid \lambda I_{p^{(\frac{(s^2+3s+2)}{2})r}-1} - [A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1} \mid = 0$ yields the characteristic equation for the adjacency matrix $[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$. Let the eigenvalues of $[A]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ be $\lambda_1, \lambda_2, \cdots, \lambda_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$. We can obtain the characteristic equation of the adjacency matrix as

$-\lambda^{p^{(\frac{(s^2+3s)}{2})r}-1}(1+\lambda)^{p^r}(\lambda^2 - 2\lambda - (p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r})) = 0.$

Upon solving for λ in each factor we obtain, $-\lambda^{p^{(\frac{(s^2+3s)}{2})r}-1} = 0$ which implies that $\lambda = 0$ of multiplicity $p^{(\frac{(s^2+3s)}{2})r} - 1$ is an eigenvalue, $(1+\lambda)^{p^r} = 0$ implies that

$\lambda = -1$ of multiplicity p^r is also an eigenvalue. By solving the quadratic equation $\lambda^2 - 2\lambda - (p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r}) = 0$ in the last factor, we obtain

$$\lambda = \frac{2 \pm \sqrt{4 + 4p^{(\frac{(s^2+3s+2)}{2})r} + 4p^{(\frac{(s^2+3s)}{2})r}}}{2} = \frac{2 \pm \sqrt{4(p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} + 1)}}{2} =$$

$$\frac{2 \pm 2\sqrt{p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} + 1}}{2} = 1 \pm \sqrt{p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} + 1}.$$

For $p \neq 2$, we generally obtain the polynomial equation which when factorized results to

$$\lambda^{2p^{(\frac{(s^2+3s)}{2})r}-1}(1+\lambda)^{p^{(\frac{(s^2+3s)}{2})r}-2}((\lambda^2 - (p^{(\frac{(s^2+3s)}{2})r} - 2)\lambda - (p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r})) = 0.$$

Solving the equation gives the eigenvalues as $\lambda^{2p^{(\frac{(s^2+3s)}{2})r}-1} = 0$ which result in $\lambda = 0$ of multiplicity $2p^{(\frac{(s^2+3s)}{2})r} - 1$. The second factor $(1+\lambda)^{p^{(\frac{(s^2+3s)}{2})r}-2} = 0$ implies that $\lambda = -1$ of multiplicity $p^{(\frac{(s^2+3s)}{2})r} - 2$. Finally, solving the quadratic part results to

$$\lambda = \frac{(p^{(\frac{(s^2+3s)}{2})r} - 2) \pm \sqrt{(p^{(\frac{(s^2+3s)}{2})r} - 2)^2 + 4(p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r})}}{2}.$$

Expanding the discriminant yields the expression

$$p^{2(\frac{(s^2+3s)}{2})r} - 4p^{(\frac{(s^2+3s)}{2})r} + 4 + 4p^{(\frac{(s^2+3s+2)}{2})r} - 4p^{(\frac{(s^2+3s)}{2})r}$$

so that

$$\lambda = \frac{(p^{(\frac{(s^2+3s)}{2})r} - 2) \pm \sqrt{p^{2(\frac{(s^2+3s)}{2})r} - 8p^{(\frac{(s^2+3s)}{2})r} + 4p^{(\frac{(s^2+3s+2)}{2})r} + 4}}{2}.$$

□

Proposition 3.4.4. *Consider* $[L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$*, the Laplacian matrix associated with* $\Gamma(R)$ *in Construction II such that* $pu_i = 0$*. Then for* $r \in \mathbb{Z}^+, p$ *prime and for a fixed* s*,*

(i) $[L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ *is symmetric.*

(ii) $rank([L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = p^{(\frac{(s^2+3s+2)}{2})r} - 2.$

(iii) $Tr([L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = 2p^{(\frac{2(s^2+3s)}{2})r} - 2p^{2(\frac{(s^2+3s)}{2})r} - 2p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r} + 2.$

(iv) $Det([L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = 0.$

(v) The eigenvalues $\lambda[L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1} = \begin{cases} 0, \\ p^{(\frac{(s^2+3s+2)}{2})r} - 1, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2})r} - 1; \\ p^{(\frac{(s^2+3s)}{2})r} - 1, & \text{of multiplicity } p^{(\frac{(s^2+3s)}{2})r} - 1. \end{cases}$

Proof. We prove (ii) to (v) as follows, (i) is clear.

(ii) Carrying out an elementary row operation on $[L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ we obtain a matrix with an echelon form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1_{p^{(\frac{(s^2+3s)}{2})r}-2} \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 & -1_{p^{(\frac{(s^2+3s)}{2})r}-1} \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & -1_{p^{(\frac{(s^2+3s)}{2})r}} \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & -1_{p^{(\frac{(s^2+3s+2)}{2})r}-3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & -1_{p^{(\frac{(s^2+3s+2)}{2})r}-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This results to $p^{(\frac{(s^2+3s+2)}{2})r} - 2$ non-zero rows in $[L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$, hence the rank.

(iii) Since $| Z(R)^* | = p^{(\frac{(s^2+3s+2)}{2})r} - 1$, each $v_i \in Ann(Z(R)^*)$ has degree $p^{(\frac{(s^2+3s+2)}{2})r} - 2$ and $| Ann(Z(R)^*) | = p^{(\frac{(s^2+3s)}{2})r} - 1$. Therefore any $v_j \notin Ann(Z(R)^*)$ is of degree $p^{(\frac{(s^2+3s)}{2})r} - 1$ because every such $v_j \notin Ann(Z(R)^*)$ is only adjacent to $v_i \in Ann(Z(R)^*)$. We partition $V \in \Gamma(R)$ into disjoint subsets V_1 and V_2 such that $V_1 = \{v_j | v_j \notin Ann(Z(R)^*)\}$ and $V_2 = \{v_i | v_i \in Ann(Z(R)^*)\}$.

Therefore, $| V_2 | = p^{(\frac{(s^2+3s)}{2})r} - 1$ and $| V_1 | = p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}$.

Since the trace, $Tr([L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = \sum_{i=1}^{p^{(\frac{(s^2+3s+2)}{2})r}-1} d_{ii}$, and every d_{ii} is an entry of the leading diagonal of the diagonal matrix $[D]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ whose diagonal are entries of the degrees of $v_i \in V(\Gamma(R))$ thus

$$Tr([L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = (p^{(\frac{(s^2+3s+2)}{2})r}-2)(p^{(\frac{(s^2+3s)}{2})r}-1)+(p^{(\frac{(s^2+3s)}{2})r}-1)(p^{(\frac{(s^2+3s+2)}{2})r}-p^{(\frac{(s^2+3s)}{2})r}).$$

Upon expansion and simplification of this equation, we obtain

$p^{(\frac{2(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 2p^{(\frac{(s^2+3s)}{2})r} + 2 + p^{(\frac{2(s^2+3s)}{2})r} - p^{(2\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} =$

$2p^{(\frac{2(s^2+3s)}{2})r} - 2p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r} - 2p^{2(\frac{(s^2+3s)}{2})r} + 2 = 2p^{(\frac{2(s^2+3s)}{2})r} - 2p^{2(\frac{(s^2+3s)}{2})r} - 2p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r} + 2.$

(iv) Simplifying $\mid [L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1} \mid = \sum_{i,j=1}^{p^{(\frac{(s^2+3s+2)}{2})r}-1} a_{ij}(-1)^{i+j} \mid l_{ij} \mid$ on the Laplacian matrix where l_{ij} are minors to $[L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ and a_{ij} are the row or column elements from the row or the column of operation, we then establish the singularity of $[L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$.

(v) Solving $\mid (\lambda I_{p^{(\frac{(s^2+3s+2)}{2})r}-1} - [L]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) \mid = 0$ gives the characteristic polynomial equation of the form;

$$-\lambda((-p^{(\frac{(s^2+3s+2)}{2})r} - 1) + \lambda)^{p^{(\frac{(s^2+3s)}{2})r}-1}(-(p^{(\frac{(s^2+3s)}{2})r} - 1) + \lambda)^{p^{(\frac{(s^2+3s)}{2})r}-1}) = 0.$$

Upon finding the value of λ in each factor, we obtain $-\lambda = 0 \implies \lambda = 0$. For the factor $(-(p^{(\frac{(s^2+3s+2)}{2})r} - 1) + \lambda)^{p^{(\frac{(s^2+3s)}{2})r}-1} = 0$, we have that $\lambda = (p^{(\frac{(s^2+3s+2)}{2})r} - 1)$ of multiplicity $p^{(\frac{(s^2+3s)}{2})r} - 1$. Finally, $(-(p^{(\frac{(s^2+3s)}{2})r} - 1) + \lambda) = 0$ implies that $\lambda = p^{(\frac{(s^2+3s)}{2})r} - 1$, this establishes (v). □

Proposition 3.4.5. *Consider $[d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$, the distance matrix associated with $\Gamma(R)$ for the ring in Construction II such that $pu_i = 0$. Then for $r \in \mathbb{Z}^+, p$, prime and s fixed,*

(i) $[d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}$ *is a singular matrix,*

(ii) $Tr([d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = 0$,

(iii) $rank([d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})r}-1}) = p^{(\frac{(s^2+3s+2)}{2})r} - 2$, *and*

(iv) Eigenvalues $$\lambda[d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})r}-1} = \begin{cases} 0, & \\ p^{(\frac{(s^2+3s+2)}{2})r}, & \\ -1, & \text{of multiplicity } p^r, \\ -p^r, & \text{of multiplicity } p^r+1. \end{cases}$$

Proof. The proofs for (i), (ii) and (iii) are clear.

(iv) Solving the equation, $\mid \lambda I - [d_{ij}]_{p^{(\frac{(s^2+3s+2)}{2})r}-1} \mid= 0$ results to the characteristic equation $-(-p^{(\frac{(s^2+3s+2)}{2})r} + \lambda)\lambda(1+\lambda)^{p^r}(p^r+\lambda)^{p^r+1} = 0$. We obtain the eigenvalues by solving the equation as follows: Clearly, $\lambda = 0$. Further, $-(-p^{(\frac{(s^2+3s+2)}{2})r} + \lambda) = 0$ implies $\lambda = p^{(\frac{(s^2+3s+2)}{2})r}$. For $(1+\lambda)^{p^r} = 0$, we have $\lambda = -1$ of multiplicity p^r. Finally, $(p^r+\lambda)^{p^r+1} = 0$ implies $\lambda = -p^r$ of multiplicity $p^r + 1$. □

Example 3.4.2. *Fix s=1, $p = 2$. Then the distance matrix associated with $\Gamma(R)$ is given by*

$$[d_{ij}] = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 0 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}_{7\times 7}.$$

Case II: when $pu_i \neq 0$

Proposition 3.4.6. *Consider R from Construction II such that $pu_i \neq 0$. Then the set $Z(R)$ of zero divisors of R satisfy the following properties;*

(i) $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$,

(ii) $(Z(R))^2 = pR' \oplus \sum_{i,j=1}^{s} R'u_iu_j$, *and*

(iii) $(Z(R))^3 = (0)$.

Proof. The proof follows a similar pattern as in the proof of Proposition 3.4.1. □

Proposition 3.4.7. *Let R be a ring from Construction II such that $pu_i \neq 0$ and $\Gamma(R)$ be its zero divisor graph. Then for $r \in \mathbb{Z}^+, p$ prime and s fixed,*

(i) $\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+5s+2)}{2})r} - 1$,

(ii) $\Gamma(R)$ *is incomplete,*

(iii) $diam(\Gamma(R)) = 2$,

(iv) $\delta(\Gamma(R)) = p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 1$, *and*

(v) $girth(\Gamma(R)) = 3$.

Proof. (i). Given that $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$ and since $\mid R' \mid = p^{2r}$, $\mid R'u_i \mid = p^{2r}$, $\mid R'u_iu_j \mid = p^r$ for $i, j = 1, 2, \cdots, s$ it implies that $\mid Z(R) \mid = p^{(\frac{(s^2+5s+2)}{2})r}$. Moreover, since $\mid Z(R)^* \mid = \mid Z(R) - \{0\} \mid$ it means that $\mid V(\Gamma(R)) \mid = \mid Z(R)^* \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1$.

(ii). Since $(Z(R))^2 \neq (0)$, it follows that not all pairs of vertices $v_i, v_j \in V(\Gamma(R))$ are connected by an edge. This shows incompleteness of $\Gamma(R)$.

(iii). There exist non adjacent vertices $v_i, v_k \in V(\Gamma(R))$ due to (ii) so that for some vertex $v_j \in Ann(Z(R)) = (Z(R))^2$, the longest path of the graph is $v_i - v_j - v_k$, which establishes (iii).

(iv). As established in (i), $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1$. Clearly $\mid Ann(Z(R)) - \{0\} \mid = \mid Ann(Z(R))^* \mid = p^{(\frac{(s^2+3s+2)}{2})r} - 1$. The minimum degree from the graph can be obtained by computing the order $\mid V(\Gamma(R)) \setminus Ann(Z(R))^* \mid =$

$$(p^{(\frac{(s^2+5s+2)}{2})r} - 1) - (p^{(\frac{(s^2+3s+2)}{2})r} - 1) = p^{(\frac{(s^2+5s+2)}{2})r} - 1 - p^{(\frac{(s^2+3s+2)}{2})r} + 1 = p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}.$$

For avoidance of self loop, we have that $\delta(\Gamma(R)) = p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 1$.

(v) Follows by part (v) of Proposition 3.3.2.

□

Proposition 3.4.8. *Consider* $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ *and* $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ *to be respectively the adjacency and Laplacian matrices for* $\Gamma(R)$ *for the ring in Construction II such that* $pu_i \neq 0$, $r \in \mathbb{Z}^+$, *p prime and s fixed. Then*

(i) $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ *and* $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ *are both symmetric,*

(ii) $rank([A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = p^{(\frac{(s^2+3s+2)}{2})r}$ *and*
$rank([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = p^{(\frac{(s^2+5s+2)}{2})r} - 2$,

(iii) $Det([A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = Det([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = 0$,

(iv) $Tr([A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = 0$ *and* $Tr([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = 2p^{(\frac{2(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{2(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} + 2,$

(v) *The eigenvalues* $\lambda[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$

$$= \begin{cases} -1, & \textit{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 2, \\ 0, & \textit{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 1, \\ (p^r+1) \pm \sqrt{\Omega} & . \end{cases}$$

where $\Omega = (p^r+1)^2 + p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(h+2)r}$, *and*

(vi) *The eigenvalues of* $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$

$$= \begin{cases} 0, & \\ p^{(\frac{(s^2+5s+2)}{2})r} - 1, & \textit{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 1, \\ (p^{(\frac{(s^2+3s+2)}{2})r} - 1), & \textit{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 1. \end{cases}$$

Proof. Proofs for Properties (i) to (iii) of $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ and $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ are obvious from definitions. We proceed to provide proof for (iv), (v) and (vi) as follows.

(iv). For the adjacency matrix, $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$, the result is clear since diagonal entries are all $0'$s. Since the order is $p^{(\frac{(s^2+5s+2)}{2})r}-1$, it follows that $\sum_{i=1}^{p^{(\frac{(s^2+5s+2)}{2})r}-1} a_{ii} = 0$ where a_{ii} are the diagonal elements of the matrix $[A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$.

We show that

$$Tr([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = 2p^{(\frac{2(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} + 2.$$

Since $\mid Z(R)^* \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1 = \mid V(\Gamma(R)) \mid$, it is established that $\mid Ann(Z(R)^*) \mid = p^{(\frac{(s^2+3s+2)}{2})r} - 1$ and any $v_i \notin Ann(Z(R)^*)$ is of degree $p^{(\frac{(s^2+3s+2)}{2})r} - 1$ since v_i is only adjacent to the vertices in $Ann(Z(R)^*)$. In the same manner, each v_j in the set $Ann(Z(R)^*)$ is connected by an edge with $v_i \in V(\Gamma(R))$. Therefore, $deg(v_j) = p^{(\frac{(s^2+5s+2)}{2})r} - 2$ for avoidance of self loop. Let the partitions of the vertex set in $\Gamma(R)$ be V_1 and V_2 such that

$V_1 = \{v_i \in Z(R)^* | v_i \notin Ann(Z(R)^*)\}$ and $V_2 = \{v_j \in Z(R)^* | v_j \in Ann(Z(R)^*)\} \Longrightarrow$ $\mid V_1 \mid = p^{(\frac{(s^2+3s+2)}{2})r} - 1$ and $\mid V_2 \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1 - (p^{(\frac{(s^2+3s+2)}{2})r} - 1) = p^{(\frac{(s^2+5s+2)}{2})r} -$

$p^{(\frac{(s^2+3s+2)}{2})r}$. Since the trace of $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$ is the sum of the diagonal entries of the degree matrix $[D]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$, that is $Tr([L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}) = \sum_{i=1}^{p^{(\frac{(s^2+5s+2)}{2})r}-1} d_{ii}$ which is equivalent to sum of degrees of the vertices in $\Gamma(R)$ where d_{ii} are the diagonal entries of $[D]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$. We have that

$(p^{(\frac{(s^2+3s+2)}{2})r} - 1)(p^{(\frac{(s^2+5s+2)}{2})r} - 2) + (p^{(\frac{(s^2+3s+2)}{2})r} - 1)(p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}) =$ $p^{(\frac{2(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} + 2 + p^{(\frac{2(s^2+5s+2)}{2})r} - p^{(\frac{2(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} +$ $p^{(\frac{(s^2+3s+2)}{2})r}$ which simplifies to $2p^{(\frac{2(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{2(s^2+3s+2)}{2})r} +$ 2. Hence the trace of $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$.

(v). Simplifying the equation $| \lambda I_{p^{(\frac{(s^2+5s+2)}{2})r}-1} - [A]_{p^{(\frac{(s^2+5s+2)}{2})r}-1} | = 0$ results to the characteristic polynomial equation

$-\lambda^{p^{(\frac{(s^2+3s+2)}{2})r}-1}(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2}(\lambda^2 - 2(p^r+1)\lambda - (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})) = 0.$

From the factorization components, $-\lambda^{p^{(\frac{(s^2+3s+2)}{2})r}-1} = 0$ which implies that $\lambda = 0$ is an eigenvalue of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 1$.

Similarly, $(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2} = 0$ implies that $\lambda = -1$ is an eigenvalue of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 2$.

For the quadratic part, $\lambda^2 - 2(p^r+1)\lambda - (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}) = 0$ results in

$$\begin{aligned}
\lambda &= \frac{2(p^r+1) \pm \sqrt{4(p^r+1)^2 + 4p^{(\frac{(s^2+5s+2)}{2})r} + 4p^{(\frac{(s^2+5s+2)}{2})r} + 4p^{(\frac{(s^2+3s+2)}{2})r}}}{2} \\
&= \frac{2(p^r+1) \pm \sqrt{4((p^r+1)^2 + p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})}}{2} \\
&= (p^r+1) \pm \sqrt{(p^r+1)^2 + p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}} \\
&= (p^r+1) \pm \sqrt{(p^r+1)^2 + p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+5s+4)}{2})r}}.
\end{aligned}$$

(vi). From the characteristic equation $| \lambda I_{p^{(\frac{(s^2+5s+2)}{2})r}-1} - [L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1} | = 0$, we obtain

$$-((-(p^{(\frac{(s^2+5s+2)}{2})r} - 1) + \lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-1}(-(p^{(\frac{(s^2+3s+2)}{2})r} - 1) + \lambda)^{p^{hr}-1})\lambda = 0.$$

Solving λ for each factor results to $(-(p^{(\frac{(s^2+5s+2)}{2})r}-1)+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-1}=0 \Longrightarrow \lambda = p^{(\frac{(s^2+5s+2)}{2})r}-1$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r}-1$. Further,
$(-(p^{(\frac{(s^2+3s+2)}{2})r}-1)+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-1}=0$ means that $\lambda = p^{(\frac{(s^2+3s+2)}{2})r}-1$ is an eigenvalue with a multiplicity of $p^{(\frac{(s^2+3s+2)}{2})r}-1$, and finally, $\lambda = 0$. Hence the eigenvalues of $[L]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$.

□

Proposition 3.4.9. *Given $[d_{ij}]_{p^{(\frac{(s^2+5s+2)}{2})r}-1}$, the distance matrix associated with $\Gamma(R)$ of the ring in Construction II. The point spectrum, $\sigma_{point}([d_{ij}]_{p^{(\frac{(s^2+5s+2)}{2})r}-1})$ is described by the following eigenvalues:*

$$\lambda = \begin{cases} -1, & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r}-2; \\ -p^r, & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r}-1; \\ (p^{(\frac{(s^2+3s+2)}{2})r}+2) \pm \sqrt{p^{(\frac{2(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s+2)}{2})r}}, & \end{cases}$$

Proof. Simplifying the equation $\mid \lambda I - [d_{ij}]_{p^{(\frac{(s^2+5s+2)}{2})r}-1} \mid = 0$ results to the polynomial equation of the form
$-(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2}(p^r+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-1}(\lambda^2-(2p^{(\frac{(s^2+3s+2)}{2})r})\lambda)+(3p^{(\frac{(s^2+3s+2)}{2})r}+4))=0$.
Finding the values of λ in each factor yields $-(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2}=0$ shows that $\lambda=-1$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r}-2$. Further, $(p^r+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-1}=0$ implies that $\lambda=-p^r$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r}-1$.
For the quadratic part $\lambda^2-(2p^{(\frac{(s^2+3s+2)}{2})r}+4)\lambda+(3p^{(\frac{(s^2+3s+2)}{2})r}+4)=0$, we obtain

$$\lambda = \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm \sqrt{(2p^{(\frac{(s^2+3s+2)}{2})r}+4)^2-4(3p^{(\frac{(s^2+3s+2)}{2})r}+4)}}{2}$$

$$= \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm \sqrt{(4p^{(\frac{2(s^2+3s+2)}{2})r}+8p^{(\frac{(s^2+3s+2)}{2})r}+8p^{(\frac{(s^2+3s+2)}{2})r}+16-12p^{(\frac{(s^2+3s+2)}{2})r}-16)}}{2}$$

$$= \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm \sqrt{4p^{(\frac{2(s^2+3s+2)}{2})r}+16p^{(\frac{(s^2+3s+2)}{2})r}-12p^{(\frac{(s^2+3s+2)}{2})r}}}{2}$$

$$= \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm \sqrt{4p^{(\frac{2(s^2+3s+2)}{2})r}+4p^{(\frac{(s^2+3s+2)}{2})r}}}{2}$$

$$= \frac{(2p^{(\frac{(s^2+3s+2)}{2})r}+4) \pm 2\sqrt{p^{(\frac{2(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s+2)}{2})r}}}{2}$$

$$= (p^{(\frac{(s^2+3s+2)}{2})r} + 2) \pm \sqrt{p^{(\frac{2(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}}.$$

□

3.5 The 3-Radical Zero Finite Completely Primary Rings of Characteristic p^3

3.5.1 Construction III

Let $R' = GR(p^{3r}, p^3)$ be a Galois ring of order p^{3r} and characteristic p^3. Let U and V be finitely generated R'-modules with the generating sets $\{u_1, u_2, \cdots, u_s\}$ and $\{v_1, v_2, \cdots, v_t\}$ respectively such that s and t are the number of elements in the generating sets. Suppose $t = \frac{s(s+1)}{2}$ for a fixed s, $R = R' \oplus \sum_{i=1}^{s} R'u_i \oplus \sum_{i,j=1}^{s} R'u_iu_j$ is an additive abelian group. Define multiplication on R by

$$(x_\circ + \sum_{i=1}^{s} x_iu_i + \sum_{i,j=1}^{s} x_ju_iu_j)(y_\circ + \sum_{i=1}^{s} y_iu_i + \sum_{i,j=1}^{s} y_ju_iu_j) =$$

$$x_\circ y_\circ + \sum_{i=1}^{s}((x_\circ + pR')y_i + x_i(y_\circ + pR')^{\sigma_i})u_i + \sum_{i,j=1}^{s}(x_\circ y_j + x_j(y_\circ)^{\sigma_i} + \sum_{i,j=1}^{s} a_{ij}x_i(y_j)^{\sigma_i})u_iu_j.$$

The multiplication given turns R into a commutative ring with identity $(1, 0, \cdots, 0, \bar{0}, \cdots, \bar{0})$ if $\sigma_i = id_{\mathbb{F}}$. From this multiplication, the set $Z(R)$ of zero divisors satisfies the following properties;

(i) $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i + \sum_{i,j=1}^{s} R'u_iu_j$,

(ii) $(Z(R))^2 = p^2R' \oplus \sum_{i,j=1}^{s} R'u_iu_j$, and

(iii) $(Z(R))^3 = (0)$.

For the rings considered in this section, $\sigma_i = id_{\mathbb{F}}$.

3.5.2 The Graphs $\Gamma(R)$ and their Matrices from Classes of Rings in Construction III

Proposition 3.5.1. *Given R, the ring of Construction III and $\Gamma(R)$ be the associated zero divisor graph. Then for any prime integer $p, r \in \mathbb{Z}^+$ and s-fixed, we have;*

(i) $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+5s+4)}{2})r} - 1$,

(ii) $\Delta(\Gamma(R)) = p^{(\frac{(s^2+5s+4)}{2})r} - 2$ *and* $\delta(\Gamma(R)) = p^{(\frac{(s^2+5s)}{2})r}$,

(iii) $\Gamma(R)$ *is incomplete,*

(iv) $diam(\Gamma(R)) = 2$, *and*

(v) $girth(\Gamma(R)) = 3$.

Proof. (i) Since $Char(R) = p^3$, $\mid R' \mid= p^{3r}$ and $\mid pR' \mid= p^{2r}$. Consider $pu_i = 0$ for $i = 1, 2, \cdots, s$ and $\mid R'u_iu_j \mid= p^r, i, j = 1, 2, \cdots, s$, we have that $\mid Z(R) \mid= p^{(\frac{(s^2+5s+4)}{2})r}$ and $\mid Z(R) \setminus \{0\} \mid=\mid (Z(R)^*) \mid=\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+5s+4)}{2})r} - 1$.

(ii) Let $\gamma_1, \gamma_2, \cdots, \gamma_r \in R'$ with $\gamma_1 = 1$ such that $\overline{\gamma}_1, \cdots, \overline{\gamma}_r \in R'$ forms a basis for R' over its prime subfield R'/pR'. From the multiplication defined on R,
$Ann(Z(R)) = \{p^2r_\circ + \sum_{i=1}^{s} a_i\gamma_iu_i + \sum_{i=1}^{s} b_i\gamma_iu_iu_j | a_i, b_i \in R', a_i + b_i \cong 0(mod\ p)\}$.
With the fact that $\mid V(\Gamma(R)) \mid= p^{(\frac{(s^2+5s+4)}{2})r} - 1$, any vertex $v_i \in Ann(Z(R))^*$ is of degree $p^{(\frac{(s^2+5s+4)}{2})r} - 2$ due to avoidance of self loop. Hence the maximum degree $\Delta(\Gamma(R))$.

Partitioning $V(\Gamma(R))$ into disjoint subsets V_1 and V_2 such that
$V_1 = \{v_i | v_i \in Ann(Z(R))^*\}$ and $V_2 = \{v_j | v_j \notin Ann(Z(R))^*\}$, $\mid V_1 \mid= p^{(\frac{(s^2+5s)}{2})r}$. This implies that the vertices of minimum degree are only adjacent to $v_i \in Ann(Z(R))^*$ and since $\mid V_1 \mid= p^{(\frac{(s^2+5s)}{2})r}$, $\delta(\Gamma(R)) = p^{(\frac{(s^2+5s)}{2})r}$.

(iii) to (v) are clear. □

The results in the sequel describe the algebraic properties of the matrices associated with $\Gamma(R)$ of the ring in Construction III.

Proposition 3.5.2. *Given* $[A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}$ *and* $[L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}$, *the adjacency and Laplacian matrices of* $\Gamma(R)$ *respectively for a ring in Construction III. Then for a prime integer* $p, r \in \mathbb{Z}^+$ *and* s *fixed,*

(i) $Det([A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}) = Det([L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}) = 0$,

(ii) $rank([A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}) = p^{(\frac{(s^2+3s+2)}{2})r} + 2$ *and*
$rank([L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}) = p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r} + 1$,

(iii) Eigenvalues $\lambda[A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1} =$

$$\begin{cases} 0, & \text{of multiplicity } p^{(\frac{(s^2+5s+2)}{2})r} + p^r + 1; \\ -1, & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 2; \\ -p^{(\frac{(s^2+3s+2)}{2})r}, & \text{of multiplicity } p^r; \\ p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1 \pm \rho. \end{cases}$$

where

$$\rho = \sqrt{(p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1)^2 - (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})},$$

and

(iv) Eigenvalues $\lambda[L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1} =$

$$\begin{cases} 0, & \text{of multiplicity } p^{(\frac{(s^2+3s+2)}{2})r} - 2; \\ 1, & \text{of multiplicity } p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} + 1; \\ 1 - p^{(\frac{(s^2+3s+2)}{2})r}, & \text{of multiplicity } p^r; \\ p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} \pm \epsilon. \end{cases}$$

where

$$\epsilon = \sqrt{(p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r})^2 - (p^{(\frac{(s^2+5s+4)}{2})r} + 2p^{(\frac{(s^2+5s+2)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} - 1)}.$$

Proof. We provide proofs for (iii) and (iv) since the steps for proofs in (i) and (ii) follow a similar pattern as in the previous section.

(iii) Expanding the characteristic equation $\mid \lambda I - [A]_{p^{(\frac{(s^2+5s+4)}{2})r}-1} \mid = 0$, we obtain the characteristic polynomial equation of the form
$-\lambda^{(p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r}+1)}(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2}(p^{(\frac{(s^2+3s+2)}{2})r} + \lambda)^{p^r}(\lambda^2 - 2(p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1)\lambda + (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})) = 0.$

Finding the value of λ from each factor in the above equation results to
$-\lambda^{p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r}+1} = 0$ which gives $\lambda = 0$ of multiplicity $p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r}+1$. Also,
$(1+\lambda)^{p^{(\frac{(s^2+3s+2)}{2})r}-2} = 0$ gives $\lambda = -1$ of multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 2$ and the factor $(p^{(\frac{(s^2+3s+2)}{2})r} + \lambda)^{p^r} = 0$ implies $\lambda = -p^{(\frac{(s^2+3s+2)}{2})r}$ of multiplicity p^r.

The quadratic part

$$\lambda^2 - 2(p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1)\lambda + (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r}) = 0$$

can be solved as

$$\lambda = \frac{2(p^{(\frac{(s^2+3s+2)}{2})r}+p^r+1)\pm\sqrt{4(p^{(\frac{(s^2+3s+2)}{2})r}+p^r+1)^2-4(p^{(\frac{(s^2+5s+4)}{2})r}+p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s+2)}{2})r})}}{2}.$$

$$= \frac{2(p^{(\frac{(s^2+3s+2)}{2})r}+p^r+1)\pm\sqrt{4((p^{(\frac{(s^2+3s+2)}{2})r}+p^r+1)^2-(p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s+2)}{2})r}))}}{2}.$$

Simplifying this equation yields

$$\lambda = (p^{(\frac{(s^2+3s+2)}{2})r}+p^r+1)\pm\sqrt{(p^{(\frac{(s^2+3s+2)}{2})r} + p^r + 1)^2 - (p^{(\frac{(s^2+5s+4)}{2})r} + p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s+2)}{2})r})}.$$

(iv) Similarly, we can provide proof for eigenvalues of $[L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1}$ by solving the equation $\mid \lambda I - [L]_{p^{(\frac{(s^2+5s+4)}{2})r}-1} \mid= 0$ which results to the polynomial $-((-1+\lambda)^{p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r}+1}\lambda^{p^{(\frac{(s^2+3s+2)}{2})r}-2}((p^{(\frac{(s^2+3s+2)}{2})r}-1)+\lambda)^p(\lambda^2-2(p^{(\frac{(s^2+3s+2)}{2})r}+ p^{(\frac{(s^2+3s)}{2})r})\lambda + (p^{(\frac{(s^2+5s+4)}{2})r} + 2p^{(\frac{(s^2+5s+2)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} - 1)) = 0.$

Solving the equation leads to $(-1+\lambda)^{p^{(\frac{(s^2+5s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r}+1} = 0$ gives $\lambda = 1$ of multiplicity $p^{(\frac{(s^2+5s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r} + 1$, $\lambda^{p^{(\frac{(s^2+3s+2)}{2})r}-2} = 0$ implies $\lambda = 0$ of algebraic multiplicity $p^{(\frac{(s^2+3s+2)}{2})r} - 2$. Similarly, $((p^{(\frac{(s^2+3s+2)}{2})r} - 1) + \lambda)^{p^r} = 0 \Longrightarrow \lambda = 1 - p^{(\frac{(s^2+3s+2)}{2})r}$ of multiplicity p^r. The quadratic part

$\lambda^2 - 2(p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r})\lambda + (p^{(\frac{(s^2+5s+4)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} - 1) = 0$

can be solved as follows;

$$\lambda = \frac{2(p^{(\frac{(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r})\pm\sqrt{4(p^{(\frac{(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r})^2-4(p^{(\frac{(s^2+5s+4)}{2})r}+2p^{(\frac{(s^2+5s+4)}{2})r}+2p^{(\frac{(s^2+3s+2)}{2})r}-1)}}{2}$$

$$= \frac{2(p^{(\frac{(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r})\pm2\sqrt{(p^{(\frac{(s^2+3s+2)}{2})r}+p^{(\frac{(s^2+3s)}{2})r})^2-(p^{(\frac{(s^2+5s+4)}{2})r}+2p^{(\frac{(s^2+5s+2)}{2})r}+2p^{(\frac{(s^2+3s+2)}{2})r}-1)}}{2}$$

$$= (p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r}) \pm \epsilon,$$

where

$$\epsilon = \sqrt{(p^{(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(s^2+3s)}{2})r})^2 - (p^{(\frac{(s^2+5s+4)}{2})r} + 2p^{(\frac{(s^2+5s+2)}{2})r} + 2p^{(\frac{(s^2+3s+2)}{2})r} - 1)}. \qquad \square$$

CHAPTER FOUR

4-RADICAL ZERO COMPLETELY PRIMARY FINITE RINGS

4.1 Introduction

Studies aimed at classifying the unit groups of 4-radical zero rings were advanced by Owino and Ojiema in [48] and further extended by Evgeniy in [20]. Further exposition of these finite rings was advanced by Lao *et al* in [31, 33, 34] for all characteristics of the rings under consideration where the classification of their automorphisms was done via the zero divisor graphs. The cases considered were: Automorphisms of zero divisors graphs of Galois rings, square radical zero unital finite rings and the cube radical zero completely primary finite rings.

In this chapter we characterize classes of 4-radical zero completely primary finite rings by presenting some general results on zero divisor graphs and matrices associated with them. This takes into account a generalization of geometric properties of $\Gamma(R)$ and general analysis of some algebraic properties of the matrices associated with the zero divisor graph $\Gamma(R)$ for $Char(R) = p,\ p^2,\ p^3$ and p^4.

4.2 4-Radical Zero Completely Primary finite Rings of Characteristic p

4.2.1 Construction I

This construction can also be obtained from [48]. Let $R' = GR(p^r, p)$ be a Galois ring of order p^r and characteristic p. Consider finitely generated R'-modules $U,\ V$, and W such that $dim_{R'}U = s,\ dim_{R'}V = t$ and $dim_{R'}W = \lambda$ and $s + t + \lambda = h$. Let the R' modules be generated by $\{u_1, u_2 \cdots, u_s\}$,
$\{v_1, v_2, \cdots, v_t\}$ and $\{w_1, w_2, \cdots, w_\lambda\}$ respectively so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group. Suppose $s = 1,\ t = 1$ and $\lambda = h - 2$, then
$R = R' \oplus R'u \oplus R'v \oplus \sum_{k=1}^{h-2} R'w_k$ where $pu = 0,\ pv = 0,\ pw_k = 0$ such that $1 \leq k \leq h - 2$ for any prime integer p. We define multiplication on R as follows;

$$(a_\circ, a_1, a_2, \cdots, a_h)(b_\circ, b_1, b_2, \cdots, b_h) =$$

$$(a_\circ b_\circ, a_\circ b_1 + a_1 b_\circ, a_\circ b_2 + a_2 b_\circ + a_1 b_1, a_\circ b_3 + a_3 b_\circ + a_1 b_2 + a_2 b_1, \cdots, a_\circ b_h + a_h b_\circ + a_1 b_2 + a_2 b_1).$$

As established in [48], R is turned by this multiplication into a commutative ring with identity $(1, 0, 0, \cdots, 0)$ and further, the set $Z(R)$ of zero divisors of R satisfy the following properties:

$$Z(R) = R'u \oplus R'v \oplus \sum_{k=1}^{\lambda} R'w_k,$$
$$(Z(R))^2 = R'v \oplus \sum_{k=1}^{\lambda} R'w_k,$$
$$(Z(R))^3 = \sum_{k=1}^{\lambda} R'w_k,$$
$$(Z(R))^4 = (0).$$

As a consequence, the next result in the sequel holds for $\Gamma(R)$.

Proposition 4.2.1. *Let R be a ring of Construction I. Then the zero divisor graph $\Gamma(R)$ satisfies the following properties:*

(i) The cardinality of the vertices, $\mid V(\Gamma(R)) \mid = p^{hr} - 1$,

(ii) Minimum degree, $\delta(\Gamma(R)) = p^r - 1$,

(iii) Maximum degree, $\Delta(\Gamma(R)) = p^{hr} - 2$, *and*

(iv) $\Gamma(R)$ *is incomplete.*

Proof. (i) Since $Char(R) = Char(R') = p$ and $pu_i = pv_j = pw_k = 0$, then $\mid R'u_i \mid = p^{sr}$, $\mid R'v_j \mid = p^{tr}$, $\mid R'w_k \mid = p^{\lambda r}$ implies that $\mid Z(R) \mid = p^{sr}.p^{tr}.p^{\lambda r} = p^{(s+t+\lambda)r} = p^{hr}$ so that $\mid Z(R)^* \mid = \mid V(\Gamma(R)) \mid = p^{hr} - 1$.

(ii) With the multiplication described, $Ann(Z(R)) = (Z(R))^3$. Suppose the vertex set $V_1 = Ann(Z(R)) \setminus \{0\}$, we thus have that $\mid V_1 \mid = p^r - 1$. Since there are only $p^r - 1$ vertices adjacent to every vertex then the minimum degree of a vertex is $p^r - 1$.

(iii) Since the number of vertices in $\Gamma(R)$ is $p^{hr}-1$, there exist $x \in V_1$ connected to every vertex in the graph. Therefore, the degree of x, $deg(x) = (p^{hr}-1)-1 = p^{hr}-2$ for the avoidance of self loop.

(iv) Clearly, $\delta(\Gamma(R))$ is not equal to $\Delta(\Gamma(R))$ illustrating that the vertices in $\Gamma(R)$ do not have the same degree of connectedness. That is, not every pair of vertices in $\Gamma(R)$ are connected. Further, due to the fact that $(Z(R))^2 \neq (0)$, the incompleteness of $\Gamma(R)$ follows.

□

4.2.2 Matrices of Zero Divisor Graphs of a Ring in Construction I

Proposition 4.2.2. *Let R be a ring of Construction I. The adjacency and Laplacian matrices satisfy the following properties:*

(i) $[A]_{p^{hr}-1}$ *and* $[L]_{p^{hr}-1}$ *are singular,*

(ii) $rank([A]_{p^{hr}-1}) = p^{hr} - p^{(h-1)r}$,

(iii) $rank([L]_{p^{hr}-1}) = p^{(h-1)r} + 2$,

(iv) $Tr([L]_{p^{hr}-1}) = 2p^{(h+1)r} - 3p^{hr} + p^{2(h-1)r} + 2p^r + 1$,

(v) For $[A]_{p^{hr}-1}$*, the number of real and complex eigenvalues are* $p^{(h-1)r}$ *and* $p^{hr} - p^{(h-1)r} - 1$ *respectively.*
Indeed, the real eigenvalues $\lambda[A]_{p^{hr}-1} = \begin{cases} 0, & \text{of multiplicity } p^{(h-1)r}-1; \\ -1, & \end{cases}$
and the complex eigenvalues $\lambda[A]_{p^{hr}-1} = \begin{cases} (p^{(h-1)r}-2)i, & \text{of multiplicity } p^{hr}-p^r-2; \\ (p^{(h-1)r}-1)i, & \text{of multiplicity } p^r - p^{(h-1)r}+1, \end{cases}$
and

(vi) The eigenvalues $\lambda[L]_{p^{hr}-1} = \begin{cases} 0, & \\ p^{hr}-1, & \\ p^{(h-1)r}-1, & \\ 1, & \text{of multiplicity } p^{hr}-4. \end{cases}$

Proof. (i) Given the adjacency matrix $\begin{bmatrix} 0 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 1_{p^{hr}-p^{(h-1)r}} \\ 0 & 0 & \cdots & \cdots & \cdots & 0_{p^{hr}-p^{r}} \\ \vdots & \vdots & 0 & \cdots & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0_{p^{hr}-1} \end{bmatrix}$, suppose we take row 1 as the pivot row in obtaining the determinant, let $a_{11}, a_{12}, \cdots, a_{1p^{hr}-1}$ be the elements of the first row of $[A]_{p^{hr}-1}$. Expanding the minor determinants along the first row, we notice that the matrix minors of $a_{1j}, j = 1, 2, \cdots, p^{hr} - 1$ have zero determinants. That is, $a_{11}(-1)^{1+j} \mid minor\ (a_{11}) \mid = \cdots = a_{1p^{hr}-1}(-1)^{1+(p^{hr}-1)}$ $\mid minor\ (a_{1p^{hr}-1}) \mid = 0$. Therefore $\sum_{j=1}^{p^{hr}-1}((-1)^{1+j}a_{1j} \mid minor\ (a_{1j}) \mid) = 0$, hence the determinant of $[A]_{p^{hr}-1}$. A similar argument can be extended for the Laplacian matrices $[L]_{p^{hr}-1}$. This proves the singularity for the matrices.

ii) Reducing the adjacency matrix to its echelon form by conducting a row operation on it, we obtain the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & 1 & \cdots & 1_{p^{hr}-p^{(h-1)r}} \\ 0 & 0 & \cdots & \cdots & \cdots & 0_{p^{hr}-p^{r}} \\ \vdots & \vdots & 0 & \cdots & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0_{p^{hr}-1} \end{bmatrix}.$$

Clearly, from this reduced echelon form, we obtain $p^{hr}-p^{(h-1)r}$ non zero rows spanning the matrix space. This leads to a rank of $p^{hr}-p^{(h-1)r}$ for the adjacency matrix $[A]_{p^{hr}-1}$.

(iii) Similar to (ii), the Laplacian matrix obtained can be reduced to an echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & -1 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots & -1 \\ 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1_{p^{(h-1)r}+2} \\ \vdots & 0 & \cdots & & & & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0_{p^{hr}-1} \end{bmatrix}$$

which is of order $(p^{hr} - 1) \times (p^{hr} - 1)$. This results to $p^{(h-1)r} + 2$ linearly independent vectors which span the matrix row space for the Laplacian matrix $[L]_{p^{hr}-1}$, hence its rank.

(iv) Let $\gamma_1, \cdots, \gamma_r \in R'$ with $\overline{\gamma}_1, \cdots, \overline{\gamma}_r \in R'$ form a basis for R' over its prime subfield R'/pR'. From the multiplication defined on R, $Ann(Z(R)) = (Z(R))^3 = p^3R'$. Let V_1, V_2 and V_3 be the vertex sets partitioning $V(\Gamma(R))$ such that $V_1 = Ann(Z(R)^*)$. This implies that $\mid V_1 \mid = p^r - 1$. Therefore, for $x \in V_1$, $deg(x) = p^{hr} - 2$.

Consider the vertex set $V_2 = \{\gamma_i v + \sum_{k=1}^{h-2} b\gamma_i w_k | b \in R'\}$. Then, $\mid V_2 \mid = p^{(h-1)r} - p^r$ and each vertex $y \in V_2$ is adjacent to a vertex of the form $\gamma_i v + \sum_{k=1}^{h-2} \gamma_i w_k$. Therefore, $deg(y) = p^{(h-1)r} - 1$. Let the set $V_3 = \{\gamma_i u + a\gamma_i v + \sum_{k=1}^{h-2} c\gamma_i w_k | a, c \in R'\}$. This means that $\mid V_3 \mid = p^{hr} - p^{(h-1)r}$ and $deg(z) = p^r - 1$ where $z \in V_3$ and z is only adjacent to the vertices in the annihilator set V_1.

The trace of the Laplacian matrix is the sum of diagonal entries in the degree matrix $[D]_{p^{hr}-1}$. Thus,

$Tr([L]_{p^{hr}-1}) = (p^{hr} - 2)(p^r - 1) + (p^{(h-1)r} - 1)(p^{(h-1)r} - p^r) + (p^{hr} - p^{(h-1)r})(p^r - 1)$.

Upon expansion and simplification of this expression, we obtain

$$Tr([L]_{p^{hr}-1}) = 2p^{(h+1)r} - 3p^{hr} + p^{2(h-1)r} + 2p^r + 1.$$

(v) Solving the equation $\mid \lambda I - A \mid = 0$, we obtain the characteristic polynomial equation $\lambda^{p^{hr}-1} - (p^{hr} - 1)\lambda^{p^{hr}-p^r-1} - p^r\lambda^{p^{(h-1)r}} + p^{(h-1)r}\lambda^{p^{(h-1)r}-1} = 0$ which can be expressed in factor form as $\lambda^{p^{(h-1)r}-1}(1+\lambda)(\lambda^{p^{(h-1)r}-1} - \lambda^{p^r} - (p^{hr} - p^{(h-1)r})\lambda + p^{(h-1)r}) = 0$. Finding λ, we solve $\lambda^{p^{(h-1)r}-1} = 0$ to get $\lambda = 0$ of multiplicity $p^{(h-1)r} - 1$ and $(1 + \lambda) = 0$ gives $\lambda = -1$. The order of the real eigenvalues is obtained by adding the multiplicities $(p^{(h-1)r} - 1) + 1 = p^{(h-1)r}$.

The equation $(\lambda^{p^{(h-1)r}-1} - \lambda^{p^r} - (p^{hr} - p^{(h-1)r})\lambda + p^{hr}) = 0$ yields the complex eigenvalues as $(p^{(h-1)r} - 2)i$ of multiplicity $p^{hr} - p^r - 2$ and $(p^{(h-1)r} - 1)i$ of multiplicity $p^r - p^{(h-1)r} + 1$. Therefore, the sum of multiplicities of complex eigenvalues are

$$(p^{hr} - p^r - 2) + p^r - p^{(h-1)r} + 1 = p^{hr} - p^{(h-1)r} - 1.$$

(vi) For the Laplacian matrix $[L]_{p^{hr}-1}$, we evaluate $\mid \lambda I - [L]_{p^{hr}-1} \mid = 0$ to obtain the characteristic polynomial equation

$$-\lambda((-(p^{hr}-1)+\lambda)(-(p^{(h-1)r}-1)+\lambda)(-1+\lambda)^{p^{hr}-4})=0.$$

Finding the values of λ in each factor, we have $-\lambda=0$ giving $\lambda=0$. Next, $-(p^{hr}-1)+\lambda=0$ gives $\lambda=p^{hr}-1$ and further $-(p^{(h-1)r}-1)+\lambda=0$ results in $\lambda=p^{(h-1)r}-1$. Finally, $(-1+\lambda)^{p^{hr}-4}=0$ implies $\lambda=1$ is an eigenvalue of multiplicity $p^{hr}-4$. □

Proposition 4.2.3. *Let R be a ring of Construction I and $[d_{ij}]$ be the distance matrix then:*

(i) $Tr([d_{ij}])=0$,

(ii) $rank([d_{ij}])=p^{hr}-1$,

(iii) The eigenvalues $\lambda[d_{ij}]=\begin{cases}-1, & \textit{of multiplicity } p^r-1;\\ -p^r, & \textit{of multiplicity } p^{hr}-2p^r+1;\\ -(p^r-1)i, & \textit{of multiplicity } p^r-1, \textit{ where } \lambda\in\mathbb{C},\end{cases}$

and

(iv) $Det([d_{ij}])=p^{(2h+1)r}$.

Proof. (i) Since the distance between a vertex and itself $d(v_i,v_i)=0$, it means that every entry d_{ii} of $[d_{ij}]$ is zero and thus $\sum_{i=1}^{p^{hr}-1}d_{ii}=0$. Hence the trace, $Tr([d_{ij}])=0$.

(ii) We carry out an elementary row operation on $[d_{ij}]$ to obtain a row reduced matrix of the form

$$\begin{pmatrix}1 & 0 & \cdots & \cdots & 0\\ 0 & 1 & 0 & \cdots & \vdots\\ \vdots & 0 & \ddots & 0 & \\ \vdots & & & \ddots & 0\\ 0 & \cdots & \cdots & & 1_{p^{hr}-1}\end{pmatrix}.$$

Clearly there are $p^{hr}-1$ linearly independent vectors in the matrix span hence the rank.

(iii) To find the eigenvalues, we solve $\mid \lambda I-[d_{ij}]\mid=0$ to obtain the equation

$$-(1+\lambda)^{p^r-1}(p^r+\lambda)^{(p^{hr}-2p^r+1)}(\lambda^{p^r-1}-(p^{hr}-1)\lambda^{p^r-2}-(p^{(h+2)r}-1)\lambda-p^{(h+1)r})=0.$$

From the equation, the real eigenvalues are $-(1+\lambda)=0$ which is $\lambda=-1$ of multiplicity p^r-1 and $(p^r+\lambda)^{(p^{hr}-2p^r+1)}=0$ gives $\lambda=-p^r$ of multiplicity $p^{hr}-2p^r+1$.

Solving the equation $(\lambda^{(p^r-1)}-(-1+p^{hr})\lambda^{(p^r-2)}-(p^{(h+2)r}-1)\lambda-p^{(h+1)r})=0$ yields the complex eigenvalues as $-(p^r-1)i$ of multiplicity p^r-1.

(iv) In obtaining the determinant we evaluate $\sum_{i,j=1}^{p^{hr}-1}(d_{ij}(-1)^{i+j}\mid minor\ (d_{ij})\mid)=p^{hr}.p^{(h+1)r}=p^{(hr+hr+r)}=p^{(2h+1)r}$.

□

4.3 4-Radical Zero Finite Completely Primary Rings of Characteristic p^2

4.3.1 Construction II

Let $R'=GR(p^{2r},p^2)$ be a Galois ring of order p^{2r} and characteristic p^2. Consider R' modules U,V and W which are generated finitely by $\{u_1,\cdots,u_s\},\ \{v_1,v_2,\cdots,v_t\}$ and $\{w_1,w_2,\cdots,w_\lambda\}$ respectively so that $R=R'\oplus U\oplus V\oplus W$ is additive abelian group and $s+t+\lambda=h$. Assume $s=h-1,\ t=1$ and $\lambda=0$ so that
$R=R'\oplus\sum_{i=1}^{h-1}R'u_i\oplus R'v$ where $pu_i\neq 0,\ p^2u_i=0$ and $pv=0$ with $1\leq i\leq s$. The following defines multiplication on R.

$$(a_\circ,a_1,a_2,,\cdots,a_{h-1},\bar{a}_h)(b_\circ,b_1,b_2,,\cdots,b_{h-1},\bar{b}_h)=$$

$$(a_\circ b_\circ+p\sum_{i,j=1}^{h-1}a_ib_j,a_\circ b_1+a_1b_\circ,\cdots,a_\circ b_{h-1}+a_{h-1}b_\circ,a_\circ\bar{b}_h+\bar{a}_hb_\circ)$$

where $\bar{a}_h,\ \bar{b}_h\in R'/pR'$. The multiplication so defined turns R into a commutative finite ring of identity $(1,0,0,\cdots,\bar{0})$ as verified in [15]. The set $Z(R)$ satisfies the following properties;

$$Z(R)=pR'\oplus\sum_{i=1}^{s}R'u_i\oplus R'v,$$

$$(Z(R))^2=pR'\oplus p\sum_{i=1}^{s}R'u_i\oplus R'v,$$

$$(Z(R))^3=p\sum_{i=1}^{s}R'u_i,\ (Z(R))^4=(0).$$

The following result describes some properties of $\Gamma(R)$ of the ring constructed in this section.

Proposition 4.3.1. *Let R be a ring of Construction II. Then:*

(i) $\mid V(\Gamma(R)) \mid = p^{2hr} - 1$,

(ii) Maximumn degree, $\Delta(\Gamma(R)) = p^{2hr} - 2$,

(iii) $\Gamma(R)$ *is an incomplete graph, and*

(iv) Minimum degree, $\delta(\Gamma(R)) = p^{hr} - 1$.

Proof. (i) Given that the structure of zero divisors is given by $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus R'v$ and since $pu_i \neq 0$, $p^2u_i = 0$ and $pv = 0$ with $1 \leq i \leq s$, $\mid pR' \mid = p^r$, $\mid R'u_i \mid = p^{2r}$ and $\mid R'v \mid = p^r$. Therefore, $\mid Z(R) \mid = p^r(p^{2r(h-1)})p^r = p^{2hr}$. Since $\mid Z(R)^* \mid = \mid Z(R)\backslash\{0\} \mid$, $\mid Z(R)^* \mid = p^{2hr} - 1 = \mid V(\Gamma(R)) \mid$.
(ii) Let $\gamma_1, \cdots, \gamma_r \in R'$ with $\gamma_1 = 1$ such that $\overline{\gamma}_1, \cdots, \overline{\gamma}_r \in R'$ is a basis for R' over its prime subfield R'/pR'. Let $V_1 = Ann(Z(R)) \setminus \{0\}$. From the multiplication described, $Ann(Z(R)) = \{pc_1\gamma_i u_1 + \cdots + pc_{h-1}\gamma_i u_{h-1} + b\gamma_i v | c_1, \cdots, c_{h-1}, b \in R'\}$. Vertices in V_1 are adjacent to every vertex in $\Gamma(R)$. Therefore, every $y \in V_1$ is of degree $p^{2hr} - 2$ for avoidance of self loop. Hence the maximum degree $\Delta(\Gamma(R)) = p^{2hr} - 2$.
(iii) This is clear due to the fact that $(Z(R))^2 \neq (0)$ and there are at least two vertices of different degrees.
(iv) Let V_1 be the set described in (ii) and $y \in V_1$ then $deg\ y = p^{2hr} - 2$ and $\mid V_1 \mid = p^{hr} - 1$. Any vertex of minimum degree is not adjacent to any other vertex in $V(\Gamma(R))$ a part from the vertices in the set V_1. Since there are $p^{hr} - 1$ vertices in set V_1, it implies that $\delta(\Gamma(R)) = p^{hr} - 1$.

□

4.3.2 Matrices of the Zero Divisor Graph of the Ring in Construction II

The results below describe the properties of the matrices associated with $\Gamma(R)$ of the ring constructed in this section.

Proposition 4.3.2. *Let R be a ring of Construction II. Suppose $[A]_{p^{2hr}-1}$ and $[L]_{p^{2hr}-1}$ are the adjacency and Laplacian matrices respectively, then;*

(i) Both matrices are singular,

(ii) $rank([A]_{p^{2hr}-1}) = p^{2hr} - p^{hr}$,

(iii) $rank([L]_{p^{2hr}-1}) = p^{hr} + 2$,

(iv) The number of real and complex eigenvalues λ for $[A]_{p^{2hr}-1}$ is $\begin{cases} p^{hr}, & \lambda \in \mathbb{R}; \\ p^{2hr} - p^{hr} - 1, & \lambda \in \mathbb{C}. \end{cases}$

(v) The eigenvalues $\lambda[L]_{p^{2hr}-1} = \begin{cases} 0, & \\ p^{2hr} - p^{hr}, & \\ p^{hr} + p^{r}, & \\ 1, & \textit{of multiplicity } p^{hr}, \end{cases}$ *and*

(vi) $Tr([L]_{p^{2hr}-1}) = p^{2hr} + p^{hr} + p^{r}$.

Proof. The proofs for (i), (ii) and (iii) can easily be followed from Proposition 4.2.2 (i), (ii) and (iii) respectively.

(iv) Solving the equation $\mid \lambda I - [A]_{p^{2hr}-1} \mid = 0$ results to a characteristic equation of the form $\lambda^{p^{2hr}-1} - (p^{2hr} - 1)\lambda^{p^{2hr}-p^{hr}-1} - p^{hr}\lambda^{p^{hr}} + p^{hr} = 0$ which factorizes to $\lambda^{p^{hr}-1}(1 + \lambda)(\lambda^{p^{hr}-1} - \lambda^{p^{r}} - (p^{2hr} - p^{hr})\lambda + p^{hr}) = 0$. Finding the values of λ from the equation, we obtain $\lambda^{p^{hr}-1} = 0$ resulting in $\lambda = 0$ of multiplicity $p^{hr} - 1$ and $\lambda + 1 = 0 \implies \lambda = -1$, as the real eigenvalues. Therefore, by evaluating the sum of the multiplicities of real eigenvalues, we obtain the number of real eigenvalues to be $p^{hr} - 1 + 1 = p^{hr}$.

The equation from the remaining factor, $(\lambda^{p^{hr}-1} - \lambda^{p^{r}} - (p^{2hr} - p^{hr})\lambda + p^{hr}) = 0$ yields $(p^{2hr} - 1) - p^{hr} = p^{2hr} - p^{hr} - 1$ complex eigenvalues due to the fact that the adjacency matrix $[A]_{p^{2hr}-1}$ is a square matrix with $p^{2hr} - 1$ rows and columns.

(v). For the Laplacian matrix $[L]_{p^{2hr}-1}$, the equation $\mid \lambda I - [L]_{p^{2hr}-1} \mid = 0$ results to the characteristic polynomial equation of the form

$-\lambda(-(p^{2hr} - p^{hr}) + \lambda)(-(p^{hr} + p^{r}) + \lambda)(-1 + \lambda)^{p^{hr}} = 0$. Upon solving the equation, $-\lambda = 0$ gives $\lambda = 0$, $-(p^{2hr} - p^{hr}) + \lambda = 0$ implying that $\lambda = p^{2hr} - p^{hr}$ and

$-(p^{hr}+p^r)+\lambda=0$ gives $\lambda=p^{hr}+p^r$. Finally, $(-1+\lambda)^{p^{hr}}=0$ implies that $\lambda=1$ of multiplicity p^{hr}. Hence the eigenvalues for $[L]_{p^{2hr}-1}$.

(vi). Since trace can be computed as the sum of eigenvalues, $Tr([L]_{p^{2hr}-1})=\sum_{i=1}^{p^{2hr}-1}\lambda_i$ then $Tr([L]_{p^{2hr}-1})=0+p^{2hr}-p^{hr}+p^{hr}+p^r+1(p^{hr})=p^{2hr}+p^{hr}+p^r$ as required. $\square$

Proposition 4.3.3. *Let R be a ring of Construction II and $[d_{ij}]$, the distance matrix of $\Gamma(R)$ then;*

(i) $Tr([d_{ij}])=0$,

(ii) $rank([d_{ij}])=p^{2hr}-1$,

(iii) The eigenvalues $\lambda=\begin{cases}-1, & \text{of multiplicity } p^{(h+2)r}-2;\\ -p^r, & \text{of multiplicity } p^{(h+2)r}-1;\\ \frac{1}{2}(\sigma\pm\sqrt{\sigma^2-4\tau}) & .\end{cases}$

(iv) $Det([d_{ij}])=p^{(2h+2)r}$.

Proof. (i) Follows from the fact that $d(v_i,v_i)=0$, thus entries d_{ii} of the main diagonal are all $0'$s hence the trace.

(ii) Given the general distance matrix $[d_{ij}]_{p^{2hr}-1}=\begin{pmatrix}0&1&1&\cdots&\cdots&1\\1&0&1&1&\cdots&1\\ \vdots&&\vdots&\ddots&&\vdots\\0&0&\cdots&0&\cdots&0_{p^{hr}-1}\end{pmatrix}$, consider the set $V=\{v_1,\cdots,v_{p^{2hr}-1}\}$ consisting of vectors which are linearly independent from a row reduced echelon form of matrix $[d_{ij}]_{p^{2hr}-1}$ such that $v_1=\begin{pmatrix}1\\0\\.\\.\\.\\0\end{pmatrix}$, $v_2=\begin{pmatrix}0\\1\\0\\.\\.\\0\end{pmatrix},\cdots,v_{p^{2hr}-1}=\begin{pmatrix}0\\.\\.\\.\\0\\1\end{pmatrix}$. Clearly, the set V is of dimension $p^{2hr}-1$ equivalent to the dimension of the matrix, thus the matrix space is spanned by vectors in V. Therefore the $rank([d_{ij}])=p^{2hr}-1$.

(iii) We evaluate the equation $|\ [d_{ij}]-\lambda I\ |=0$ to obtain the characteristic polynomial $-(1+\lambda)^{p^{(h+2)r}-2}(p^r+\lambda)^{p^{(h+2)r}-1}(\lambda^2-(p^{(h-1)r}(p^{(h+2)r}-p^{hr}-1)\lambda+(2p^{(h+2)r}+2p^{hr}-$

$4)(p^{hr}+3))$. Finding λ in each factor, we solve $(p^r+\lambda)^{p^{(h+2)r}-1}=0$ which implies $\lambda=-p^r$ of multiplicity $p^{(h+2)r}-1$. Further, $-(1+\lambda)^{p^{(h+2)r}-2}=0$ gives $\lambda=-1$ with a multiplicity of $p^{(h+2)r}-2$. For the quadratic part, we solve

$$\lambda^2-(p^{(h-1)r}(p^{(h+2)r}-p^{hr}-1))\lambda+(2p^{(h+2)r}+2p^{hr}-4)(p^{hr}+3)=0.$$

If we let $(p^{(h-1)r}(p^{(h+2)r}-p^{hr}-1)\lambda=\sigma$ and $(2p^{(h+2)r}+2p^{hr}-4)(p^{hr}+3)=\tau$, we obtain $\frac{1}{2}(\sigma\pm\sqrt{\sigma^2-4\tau})$.

(iv) This follows form the proof of the determinant of distance matrix in Proposition 4.2.3. □

4.4 The 4-Radical Zero Finite Completely Primary Rings of Characteristic p^3

4.4.1 Construction III

Let $R'=GR(p^{3r},p^3)$ be a Galois ring of characteristic p^3 and of order p^{3r}. Consider finitely generated R' modules U,V and W with dimensions $s,\ t$ and λ respectively whose generating sets are $\{u_1,\cdots,u_s\},\{v_1,\cdots,v_t\}$ and $\{w_1,\cdots,w_\lambda\}$ where $s+t+\lambda=h$ so that $R=R'\oplus U\oplus V\oplus W$ is an additive abelian group. Consider $s=h-1,\ t=1$ and $\lambda=0$ so that $R=R'\oplus\sum_{i=1}^{h-1}R'u_i\oplus R'v$ where $p^2u_i\neq 0,\ p^3u_i=0$ where $1\leq i\leq s$ and $pv=0$. Multiplication is defined in R as follows:

$$(a_\circ,\overline{a}_1,\overline{a}_2,\cdots,\overline{a}_{h-1},\tilde{a}_h)(b_\circ,\overline{b}_1,\overline{b}_2,\cdots,\overline{b}_{h-1},\tilde{b}_h)=$$

$$(a_\circ b_\circ,a_\circ\overline{b}_1+\overline{a}_1b_\circ,\cdots,a_\circ\overline{b}_{h-1}+\overline{a}_{h-1}b_\circ,a_\circ\tilde{b}_h+\tilde{a}_hb_\circ+\sum_{i,j=1}^{h-1}\overline{a_ib_j})$$

where $\overline{a}_i,\overline{b}_j\in R'/p^2R'$ and $\tilde{a}_h,\tilde{b}_h\in R'/pR'$. From [45], it is verifiable that R is turned into a commutative ring with identity $(1,\overline{0},\cdots,\overline{0},\tilde{0})$ by the multiplication.

The set of zero divisors $Z(R)$ satisfy the properties below;

$$Z(R)=pR'\oplus\sum_{i=1}^{s}R'u_i\oplus R'v,$$

$$(Z(R))^2=p^2R'\oplus p\sum_{i=1}^{s}R'u_i\oplus R'v,$$

$$(Z(R))^3 = pR'v,$$

and

$$(Z(R))^4 = (0).$$

The results in the sequel describe some properties of $\Gamma(R)$ of the ring constructed in this Section.

Proposition 4.4.1. *Let R be a ring of Construction III. Then:*

(i) The cardinality, $\mid V(\Gamma(R)) \mid= p^{3hr} - 1$,

(ii) The maximum degree, $\Delta(\Gamma(R)) = p^{3hr} - 2$,

(iii) The minimum degree, $\delta(\Gamma(R)) = p^{hr} - 1$, and

(iv) The graph $\Gamma(R)$ is incomplete.

Proof. (i) Given that $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i \oplus R'v$ and that $p^2u_i \neq 0$, $p^3u_i = 0$ and $pv = 0$, it follows that $\mid pR' \mid= p^{2r}$, $\mid R'u_i \mid= p^{3r}$ and $\mid R'v \mid= p^r$. Therefore, $\mid Z(R) \mid= p^{2r}(p^{3r(h-1)})p^r = p^{3hr}$. Since $\mid Z(R) \setminus \{0\} \mid=\mid (Z(R))^* \mid= p^{3hr} - 1$, then $\mid (Z(R))^* \mid=\mid V(\Gamma(R)) \mid= p^{3hr} - 1$.

The proofs for (ii) and (iii) can be followed easily from Proposition 4.3.1 parts (ii) and (iv) respectively.

For (iv), the fact that $(Z(R))^2 \neq (0)$ explains the incompleteness of $\Gamma(R)$. □

Proposition 4.4.2. *Let R be a ring of Construction III. Suppose V_1, V_2, V_3, V_4 and V_5 are the partitions of $V(\Gamma(R))$. Then the degrees of vertices $v \in V(\Gamma(R))$*

$$= \begin{cases} p^{3hr} - 2, & v \in V_1 \text{ and } \mid V_1 \mid= p^{hr} - 1; \\ p^{2hr} - 2, & v \in V_2 \text{ and } \mid V_2 \mid= p^{2hr} - p^{hr}; \\ deg(v) \in (X \cup Y) = V_3, & v \in V_3 \text{ and } \mid V_3 \mid= p^{(h+1)r} - p^{(h-1)r}\ ; \\ deg(v) \in (W \cup Z) = V_4, & v \in V_4 \text{ and } \mid V_4 \mid= 2p^{(h+2)r}; \\ p^{(h+1)r} - p^{hr} + p^{(h-1)r} - 1, & v \in V_5 \text{ and } \mid V_5 \mid= p^{3hr} - 2p^{(h+2)r} + p^{(h+1)r}. \end{cases}$$

Proof. We describe the connectedness of $\Gamma(R)$ as follows:

Let $\gamma_1, \cdots, \gamma_r \in R'$ with $\gamma_1 = 1$ such that $\overline{\gamma}_1, \cdots, \overline{\gamma}_r \in R'$ is the basis of R' over its prime subfield R'/pR'. From the defined multiplication, $Ann(Z(R)) = \{p^2\gamma_i u_1 + \cdots +$

$p^2\gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\}$. Let $V_1 = Ann(Z(R))^*$, therefore the order of V_1, $\mid V_1 \mid = p^{hr} - 1$. Every $v \in V_1$ is adjacent to each vertex in $\Gamma(R)$ and therefore the degree, $deg(v) = p^{3hr} - 2$.

Similarly, consider set $V_2 = \{p^2 r_\circ + p^2\gamma_i u_1 + \cdots + p^2\gamma_i u_{h-1} + b\gamma_i v \mid p^2 r_\circ \neq 0, b \in R'\}$. Each vertex $v \in V_2$ is connected to other vertices in $\Gamma(R)$ apart from the vertices of the form $pr_\circ + \gamma_i u_1 + \cdots + \gamma_i u_{h-1} + b\gamma_i v, b \in R'$ where $r_\circ$ is not a multiple of p. Thus, $\mid V_2 \mid = p^{2hr} - p^{hr}$ and $deg(v) = p^{2hr} - 2$ for every $v \in V_2$.

Next, suppose $X = \{p^2 r_\circ + p\gamma_i u_1 + \cdots + p\gamma_i u_{h-1} + b\gamma_i v\} \setminus V_1 \cup V_2$. It means that the order of X, $\mid X \mid = p^{(h+1)r} - p^{hr}$. Each vertex in set X is connected to a vertex in either set V_1, V_2, X or Y where $Y = \{p\gamma_i u_1 + \cdots + p\gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\} \setminus V_1$. This implies that $\mid Y \mid = p^{hr} - p^{(h-1)r}$ hence, $deg(v) = p^{(h-1)r} - 1 + p^{hr} - p^{(h-1)r} + p^{hr} - p^{(h-1)r} + p^{(h+1)r} - p^{hr} - 1 = p^{(h+1)r} + p^{hr} - 2p^{(h-1)r} - 2$ for every $v \in X$ and each $v \in Y$ is adjacent to either a vertex in V_1, V_2, X or Y. Thus $deg(v) = p^{(h+1)r} + p^{hr} - 2p^{(h-1)r} - 2$ for every $v \in Y$.

Further, let $V_3 = X \cup Y$. and consider set $W = \{pr_\circ + p\gamma_i u_1 + \cdots + p\gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\} \setminus V_1 \cup V_2 \cup V_3$. Therefore, the order of W, $\mid W \mid = p^{(h+2)r} - (p^{(h-1)r} + p^{(h+1)r} - p^{hr} + p^{hr} - p^{(h-1)r}) = p^{(h+2)r} - p^{(h+1)r}$. Each $v \in W$ is either adjacent to a vertex in V_1 or V_2 therefore, $deg(v) = p^{(h-1)r} - 1 + p^{hr} - p^{(h-1)r} = p^{hr} - 1$ for every $v \in W$.

Similarly, let $Z = \{p^2 r_\circ + \gamma_i u_1 + \cdots + \gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\}$. It means that the order of Z, $\mid Z \mid = p^r(p^{hr} - p^{(h-1)r})p^r = p^{(h+2)r} - p^{(h+1)r}$. Each vertex, $v \in Z$ is either connected to a vertex in V_1 or Y. So, $deg(v) = p^{(h-1)r} - 1 + p^{hr} - p^{(h-1)r} = p^{hr} - 1$. We finally consider the set $V_4 = W \cup Z$. and let set $V_5 = \{pr_\circ + \gamma_i u_1 + \cdots + \gamma_i u_{h-1} + b\gamma_i v \mid b \in R'\} \setminus Z$. Then, $\mid V_5 \mid = p^{(h-1)r}(p^{(h-1)r})p^r - (p^{(h+2)r} - p^{(h+1)r} = p^{(h-1)r}(p^{(h+1)r} - p^{hr}) - (p^{(h+2)r} - p^{(h+1)r}) = p^{(h+3)r} - 2p^{(h+2)r} + p^{(h+1)r}$. Therefore the degree of every vertex in V_5 is $p^{(h-1)r} - 1 + (p^{(h+1)r} - p^{hr}) = p^{(h+1)r} - p^{hr} + p^{(h-1)r} - 1$.

□

4.4.2 Matrices of the Zero Divisor Graph of a Ring in Construction III

The following results describe some properties of the adjacency, Laplacian and distance matrices associated with $\Gamma(R)$ of the ring described in Construction III.

Proposition 4.4.3. *Let R be a ring of Construction III. The adjacency and Laplacian matrices have the following properties;*

(i) $[A]_{p^{3hr}-1}$ and $[L]_{p^{3hr}-1}$ are both singular and symmetric.

(ii) $rank([A]_{p^{3hr}-1}) = p^{3hr} - p^{2hr} + p^r + 1$.

(iii) $rank([L]_{p^{3hr}-1}) = p^{3hr} - p^{(h-1)r}$.

(iv) The number of real and complex eigenvalues λ is $\begin{cases} p^{2hr} - p^{hr} + 1, & \lambda \in \mathbb{R}; \\ p^{3hr} - p^{2hr} - p^{hr}, & \lambda \in \mathbb{C}. \end{cases}$ for both the adjacency and Laplacian matrices.

Proof. The steps for the proof of (i),(ii) and (iii) are similar to Proposition 4.2.2. We provide the proof for (iv) as follows.
Upon solving the equation $| \lambda I - [A]_{p^{3hr}-1} |= 0$, we obtain the real eigenvalues by evaluating $-\lambda^{(p^{2hr}-p^{hr}-p^r)}(1+\lambda)^{p^r+1} = 0$. This implies that $\lambda = 0$ of multiplicity $p^{2hr} - p^{hr} - p^r$ and $\lambda = -1$ of multiplicity $p^r + 1$. Therefore real eigenvalues are $p^{2hr} - p^{hr} - p^r + p^r + 1 = p^{2hr} - p^{hr} + 1$ in number. The number of complex eigenvalues in $[A]_{p^{3hr}-1}$ is $(p^{3hr} - 1) - (p^{2hr} - p^{hr} + 1) = p^{3hr} - p^{2hr} - p^{hr}$.
For the Laplacian matrix, simplifying $| \lambda I - [L]_{p^{3hr}-1} |= 0$ results to the characteristic equation of the form $-(-1+\lambda)^{(p^{2hr}-p^{hr}-p^r)}\lambda^{p^r+1} = 0$. Solving the equation yields real eigenvalues $\lambda = 0$ of multiplicity $p^r + 1$ and $(-1+\lambda)^{(p^{2hr}-p^r-1)} = 0$ implying that $\lambda = 1$ of multiplicity $p^{2hr} - p^{hr} - p^r$. Therefore, the number of real eigenvalues are $p^{2hr} - p^{hr} - p^r + p^r + 1 = p^{2hr} - p^{hr} + 1$. From this and given that the matrix is of order $p^{3hr} - 1$, the complex eigenvalues are $p^{3hr} - p^{2hr} - p^{hr}$ in number. □

Proposition 4.4.4. *Let R be a ring of Construction III and $[d_{ij}]$, the distance matrix. Then;*

(i) $Tr([d_{ij}]) = 0$,

(ii) $rank([d_{ij}]) = p^{2hr} - 2$,

(iii) The eigenvalues $\lambda = \begin{cases} -1, & \textit{of multiplicity } p^{2hr}; \\ -p^{2r}, & \textit{of multiplicity } p^{2hr} - 1; \quad \textit{and} \\ p^{2r} + 1 & , \end{cases}$

(iv) $Det([d_{ij}]) = p^{hr}$.

Proof. The steps for the proof are similar to those in Propositions 4.3.3. □

4.5 4-Radical Zero Finite Completely Primary Rings of Characteristic p^4

4.5.1 Construction IV

Let $R' = GR(p^{4r}, p^4)$ be a Galois ring of order p^{4r} and characteristic p^4. Consider finitely generated R'-modules U, V and W generated by $\{u_1, u_2, \cdots, u_s\}, \{v_1, v_2, \cdots, v_t\}$ and $\{w_1, w_2, \cdots, w_\lambda\}$ respectively. Let $dim_{R'}U = s,\ dim_{R'}V = t$ and $dim_{R'}W = \lambda$, so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group and $s + t + \lambda = h$. Assume that $s = h,\ t = 0$ and $\lambda = 0$ so that $R = R' \oplus \sum_{i=1}^{s} R'u_i$ with $pu_i = 0,\ 0 \leq i \leq s$. The multiplication on R is defined by;

$$(a_\circ, \overline{a}_1, \cdots, \overline{a}_h)(b_\circ, \overline{b}_1, \cdots, \overline{b}_h) =$$

$$(a_\circ b_\circ, a_\circ \overline{b}_1 + \overline{a}_1 b_\circ, \cdots, a_\circ \overline{b}_h + \overline{a}_h b_\circ)$$

where $\overline{a}_i,\ \overline{b}_j \in R'/pR'$ and $1 \leq i, j \leq s$. The ring R is turned by this multiplication into a commutative ring with identity $(1, \overline{0}, \cdots, \overline{0})$. The set $Z(R)$ satisfy the following properties;

$$Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i,$$

$$(Z(R))^2 = p^2 R',$$

$$(Z(R))^3 = p^3 R',$$

and

$$(Z(R))^4 = (0).$$

The following result describes the zero divisor graph $\Gamma(R)$ of the ring constructed in this section.

Proposition 4.5.1. *Let R be a ring of Construction IV. Let V_1, V_2, V_3 and V_4 be the order of partitions of vertices in $V(\Gamma(R))$. Then:*

(i) The cardinality, $\mid V(\Gamma(R)) \mid = p^{(h+3)r} - 1$, and

$$(ii)\ deg(v) = \begin{cases} p^{(h+3)r} - 2, & v \in V_1 \text{ and } \mid V_1 \mid = p^{(h+2)r} - 1; \\ p^{hr} + p^{(h-1)r} + p^r, & v \in V_2 \text{ and } \mid V_2 \mid = p^{hr}; \\ p^{(h+1)r} - p^r, & v \in V_3 \text{ and } \mid V_3 \mid = p^{hr} + p^{(h-1)r}; \\ p^{hr} - p^{(h-1)r} + 1, & v \in V_4 \text{ and } \mid V_4 \mid = p^{(h+1)r} - p^{hr}. \end{cases}$$

Proof. (i) Given $Z(R) = pR' \oplus \sum_{i=1}^{s} R'u_i$ and that $pu_i = 0$, then, $\mid Z(R) \mid = \mid V(\Gamma(R)) \mid$. Further, $\mid pR' \mid = p^{3r}$ and $\mid R'u \mid = p^{hr}$. Therefore, $\mid Z(R) \mid = p^{3r}(p^{hr}) = p^{(h+3)r}$ and $\mid Z(R) \setminus \{0\} \mid = p^{(h+3)r} - 1 = \mid V(\Gamma(R)) \mid$.

(ii) Let $\gamma_1, \gamma_2, \cdots, \gamma_r \in R'$ with $\gamma_1 = 1$ such that $\overline{\gamma}_1, \overline{\gamma}_2, \cdots \overline{\gamma}_r \in R'$ forms a basis of R' over its prime subfield R'/pR'. From the multiplication given, $Ann(Z(R)) = \{p^3 r_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h \mid b \in R'\}$. Let $V_1 = Ann(Z(R)) \setminus \{0\}$. This implies that $\mid V_1 \mid = p^{(h-1)r} - 1$. Each vertex $v \in V_1$ is connected to every other vertex in $V(\Gamma(R))$. Therefore, $deg(v)$ in the set V_1 is $p^{(h+3)r} - 1 - 1 = p^{(h+3)r} - 2$.

Let $V_2 = \{p^3 r_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h \mid b \in R'\}$. Clearly $\mid V_2 \mid = p^{hr}$ and every $v \in V_2$ is adjacent to a vertex of the form $pr_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h$ therefore, $deg(v)$ in the set V_2 is $p^{hr} + p^{(h-1)r} + p^r$.

Further, let $V_3 = \{p^2 r_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h \mid b \in R'\}$ then $\mid V_3 \mid = p^{hr} + p^{(h-1)r}$. Each $v \in V_3$ is adjacent to the vertex of the form $p^2 r_\circ + b\gamma_i u_1 + \cdots + b\gamma_i u_h$. Therefore, $deg(v)$ in V_3 is $p^{(h+1)r} - p^r$.

Finally, let $V_4 = \{pr_\circ + \gamma_i u_1 + \cdots + \gamma_i u_h\} \setminus V_1 \cup V_3$, then $\mid V_4 \mid = p^{(h+1)r} - (p^{(h-1)r} + p^{hr} - p^{(h-1)r}) = p^{(h+1)r} - p^{hr}$. Each $v \in V_4$ is either adjacent to a vertex in V_1 or V_2. So, $deg(v)$ in the set V_4 is $p^{hr} - (p^{(h-1)r} - 1) = p^{hr} - p^{(h-1)r} + 1$. □

4.5.2 Matrices of the Zero Divisor Graph of a Ring in Construction IV

Proposition 4.5.2. *Let R be a ring of Construction IV. The adjacency and Laplacian matrices satisfy the following properties;*

(i) $[A]_{p^{(h+3)r}-1}$ *and* $[L]_{p^{(h+3)r}-1}$ *are both singular,*

(ii) $rank([A]_{p^{(h+3)r}-1}) = p^{hr} + p^{(h-2)r} + 1$,

(iii) $rank([L]_{p^{(h+3)r}-1}) = p^{(h+1)r} + p^{hr} + 2$, *and*

(iv) The number of real and complex eigenvalues λ *is* $\begin{cases} p^{(h+1)r} + 2p^{(h-1)r}, & \lambda \in \mathbb{R}; \\ p^{(h+2)r} - p^{(h+1)r} - 2p^{(h-1)r} - 1, & \lambda \in \mathbb{C}. \end{cases}$

for both $[A]_{p^{(h+3)r}-1}$ *and* $[L]_{p^{(h+3)r}-1}$.

Proof. We provide a proof for (iv). The proofs for (i), (ii) and (iii) are clear.

Upon obtaining the characteristic polynomial for the adjacency matrix, we find the real eigenvalues from the equation $-\lambda^{(p^{(h+1)r}+p^{(h-1)r}-1)}(1+\lambda)^{(p^{(h-1)r}+1)} = 0$.

The solution to this result is $\lambda = 0$ of multiplicity $p^{(h+1)r} + p^{(h-1)r} - 1$ and $(1+\lambda)^{(p^{(h-1)r}+1)} = 0$ implying that $\lambda = -1$ of multiplicity $p^{(h-1)r} + 1$. Therefore, the number of real eigenvalues from the characteristic polynomial equation of the adjacency matrix is $p^{(h+1)r} + p^{(h-1)r} + p^{(h-1)r} - 1 + 1 = p^{(h+1)r} + 2p^{(h-1)r}$.

Given that the adjacency matrix $[A]_{p^{(h+3)r}-1}$ is a square matrix with $p^{(h+2)r} - 1$ rows and columns and its characteristic polynomial has both real and complex parts, we have that the number of complex eigenvalues are $(p^{(h+2)r} - 1) - p^{(h+1)r} + 2p^{(h-1)r} = p^{(h+2)r} - p^{(h+1)r} - 2p^{(h-1)r} - 1$.

For the Laplacian matrix, the characteristic polynomial equation is of the form $-(\lambda^{p^{(h+2)r}-1} + \lambda p^{(h+1)r} - p^{(h-1)r})(-1+\lambda)^{(p^{(h+1)r}+p^{(h-1)r}-1)}\lambda^{(p^{(h-1)r}+1)} = 0$. From the equation, we obtain the real eigenvalues by solving $(-1+\lambda)^{(p^{(h+1)r}+p^{(h-1)r}-1)}\lambda^{(p^{(h-1)r}+1)} = 0$. This implies that $\lambda = 0$ of multiplicity $p^{(h-1)r} + 1$ and $\lambda = 1$ of multiplicity $p^{(h+1)r} + p^{(h-1)r} - 1$. Similarly, we can find the values of λ in the remaining factor by solving the equation $-(\lambda^{p^{(h+2)r}-1} + \lambda p^{(h+1)r} - p^{(h-1)r}) = 0$ to obtain the complex eigenvalues. □

CHAPTER FIVE

SOME GRAPH INDICES AND GENERAL MATRIX PROPERTIES OF THE CLASSES OF COMPLETELY PRIMARY FINITE RINGS

5.1 Introduction

In the previous chapters, we discussed the zero divisor graphs, matrices of the graphs and the algebraic properties of these matrices. We take a further discussion on properties resulting from either the algebraic properties, spectral properties and some matrix properties from the subgraphs of these graphs in Propositions 5.1.1 to 5.1.4. Some general algebraic properties such as the order, trace, rank and spectral properties of matrices of the subgraphs discussed in Propositions 5.1.5 to 5.1.8 are from the induced subgraphs obtained by the removal of a vertex with minimum or maximum degree from the zero divisor graphs. We also present the graph indices such as the binding number in Propositions 5.2.1 to 5.2.4, Wiener index (Propositions 5.3.1,5.3.4 and 5.3.6), average distance indices and average disorder numbers in Propositions 5.3.2,5.3.3 and 5.3.5 together with bounds on the Zagreb indices of $\Gamma(R)$ discussed in Propositions 5.4.1, 5.4.2 and 5.4.3.

Proposition 5.1.1. *Consider the adjacency matrix* $[A]_{p^{(h+(k-1))r}-1}$ *of* $\Gamma(R)$ *of the classes of rings described by Constructions I, II and III in 4.2.1, 4.3.1 and 4.4.1 respectively. Let* p*, be prime,* $r, k \in \mathbb{Z}^+$ *and* h *be the dimension of* $R'-$*modules. Then,*

$$\prod_{i=1}^{p^{(h+(k-1))r}-1} \lambda_i = 0$$

and

$$\sum_{i=1}^{p^{(h+(k-1))r}-1} \lambda_i = Tr([A]_{p^{(h+(k-1))r}-1})$$

where λ_i *are eigenvalues of* $[A]_{p^{(h+(k-1))r}-1}$.

Proof. Let $[A]_{p^{(h+(k-1))r}-1}$ be the adjacency matrix of $\Gamma(R)$. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_{p^{(h+(k-1))r}-1}$ are the eigenvalues obtained from its characteristic polynomial equation

$p(\lambda) = \lambda^{p^{(h+(k-1))r}-1} + c_{p^{(h+(k-1))r}-2}\lambda^{p^{(h+(k-1))r}-2} + \cdots + c_1\lambda + c_\circ$ where $c_1, c_2, \cdots, c_{p^{(h+(k-1))r}-2}$ are constant coefficients. From the Cayley Hamilton Theorem, we have that $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_{p^{(h+(k-1))r}-1})$. Considering $c_\circ$, the constant term in the characteristic polynomial equation, we can obtain it in two ways. First, $p(0) = (0 - \lambda_1)(0 - \lambda_2)\cdots(0 - \lambda_{p^{(h+(k-1))r}-1}) = (-1)^{p^{(h+(k-1))r}-1}\lambda_1 \cdots \lambda_{p^{(h+(k-1))r}-1}$. Alternatively, $p(0) =\mid 0I - A \mid=\mid -A \mid= (-1)^{p^{(h+(k-1))r}-1} \mid A \mid$. Therefore, $c_\circ = (-1)^{p^{(h+(k-1))r}-1}\lambda_1 \cdots \lambda_{p^{(h+(k-1))r}-1} = (-1)^{p^{(h+(k-1))r}-1} \mid A \mid$ which implies that $\lambda_1.\lambda_2.\cdots \lambda_{p^{(h+(k-1))r}-1} =\mid [A]_{p^{h+(k-1)r}-1} \mid$. As seen earlier, the adjacency matrices of class of rings in the chapter were found to be singular, the product of $p^{(h+(k-1))r} - 1$ eigenvalues of $[A]_{p^{(h+(k-1))r}-1}$ which gives determinant. Therefore, $\prod_{i=1}^{p^{(h+(k-1))r}-1} \lambda_i = 0$. Next, we show that $\sum_{i=1}^{p^{(h+(k-1))r}-1} \lambda_i = trace([A]_{p^{(h+(k-1))r}-1})$.

Consider $c_{p^{(h+(k-1))r}-1}$, the coefficient of $\lambda^{p^{(h+(k-1))r}-2}$. The coefficient can be calculated by first expanding $p(\lambda) = (\lambda - \lambda_{p^{(h+(k-1))r}-1})\cdots(\lambda - \lambda_{p^{(h+(k-1))r}-1})$. To obtain the term in $\lambda^{p^{(h+(k-1))r}-2}$, we choose λ from $p^{(h+(k-1))r} - 2$ of the factors and constant from the other. This implies that $\lambda^{p^{(h+(k-1))r}-2}$ term will be
$-\lambda_1\lambda^{p^{(h+(k-1))r}-2} - \cdots - \lambda_{p^{(h+(k-1))r}-1}\lambda^{p^{(h+(k-1))r}-2} =$
$-(\lambda_1 + \lambda_2 + \cdots + \lambda_{p^{(h+(k-1))r}-1})\lambda^{p^{(h+(k-1))r}-2} \Rightarrow c_{p^{(h+(k-1))r}-2} = -(\lambda_1 + \lambda_2 + \cdots + \lambda_{p^{(h+(k-1))r}-1})$.

Further, we can obtain the coefficient $c_{p^{(h+(k-1))r}-2}$ of $\lambda^{p^{(h+(k-1))r}-2}$ upon simplification of $\mid \lambda I - A \mid$. We thus calculate the determinant by finding the product of elements in positions $1j_1, 2j_2, 3j_3, \cdots, (p^{(h+(k-1))r} - 1)j_{p^{(h+(k-1))r}-1}$ for all possible permutations $j_1, j_2, \cdots, j_{p^{(h+(k-1))r}-1}$ of $1, 2, 3, \cdots p^{(h+(k-1))r} - 1$. The $(p^{(h+(k-1))r} - 1)!$ products are added together to give determinant. One of such products is $(\lambda - a_{11})(\lambda - a_{22})\cdots(\lambda - a_{p^{(h+(k-1))r}-1})$. Other products have $p^{(h+(k-1))r} - 3$ elements on the leading diagonal of the matrix which will contain at most $(p^{(h+(k-1))r} - 3)$ $\lambda's$. When an expansion of all these is done, a polynomial with degree at most $p^{(h+(k-1))r} - 3$ is obtained. Let this polynomial be denoted as $g(\lambda)$. Therefore, $p(\lambda) = (\lambda - a_{11})\cdots(\lambda - a_{(p^{(h+(k-1))r}-1)(p^{(h+(k-1))r}-1)}) + g(\lambda)$.

Since $deg(g(\lambda)) \leq p^{(h+(k-1))r} - 3$, it has no $\lambda^{p^{(h+(k-1))r}-2}$ term and therefore the leading term of $g(\lambda)$ must be from the product

$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{(p^{(h+(k-1))r}-1)(p^{(h+(k-1))r}-1)})$. From this argument it shows that the term has to be $-(a_{11} + \cdots + a_{(p^{(h+(k-1))r}-1)(p^{(h+(k-1))r}-1)})\lambda^{p^{(h+(k-1))r}-2}$.

So, $c_{p^{(h+(k-1))r}-2} = -(\lambda_1 + \cdots + \lambda_{p^{(h+(k-1))r}-1}) = -(a_{11} + a_{22} + \cdots + a_{(p^{(h+(k-1))r}-1)(p^{(h+(k-1))r}-1)})$.

$\lambda_1 + \lambda_2 + \cdots + \lambda_{p^{(h+(k-1))r}-1} = a_{11} + a_{22} + \cdots + a_{(p^{(h+(k-1))r}-1)(p^{(h+(k-1))r}-1)}$ is the trace of $[A]_{p^{(h+(k-1))r}-1}$.

□

Definition 5.1.1. *[63] A component of a graph $G = (V, E)$ is the maximal connected subgraph, where maximality condition mean that a subgraph $H \subseteq G$ is a connected subgraph and for any $v \in V(G)$, $v \notin V(H)$, $G[V(H) \cup \{v\}]$ is disconnected.*

Definition 5.1.2. *[63] Let G=(V,E) be a graph with a collection of vertices V and edges E. The nullity of the graph G with n vertices, m edges and k components is a non-negative integer $\eta(G) = m - n + k$ and $\eta(G)$=the multiplicity of 0 eigenvalue of the adjacency or the Laplacian matrix of G.*

Proposition 5.1.2. *Let $\Gamma(R)$ be the zero divisor graph of classes of rings described by Constructions I, II and III in subsections 3.3.1, 3.4.1 and 3.5.1 respectively and let $[A]$ be the adjacency matrix associated with $\Gamma(R)$. If $\eta(\Gamma(R))$ is the nullity of $\Gamma(R)$, then for any prime integer p and $r, s \in \mathbb{Z}^+$ with s fixed,*

$$\eta(\Gamma(R)) = \begin{cases} 2p^r - 1, & \text{when } Char(R) = p, p \neq 2; \\ p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}, & \text{when } Char(R) = p^2, pu_i = 0; \\ p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}, & \text{when } Char(R) = p^2, pu_i \neq 0; \\ p^{(\frac{(s^2+5s+4)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 3, & \text{when } Char(R) = p^3. \end{cases}$$

Proof. From the Dimension Theorem formula: $dim(V) = dim(ker(T)) + dim(im(T))$ where T is a linear transformation and V is a vector space over T, we can obtain the nullity of the zero divisor graph $\Gamma(R)$ with respect to the adjacency matrices of the classes of rings as follows:

$\eta(\Gamma(R)) = dim([A]) - rank([A])$ but $dim([A]) = \mid V(\Gamma(R)) \mid = \mid (Z(R))^* \mid$.

Case (i): When $Char(R) = p$ and $p \neq 2$,

$rank(\Gamma(R)) = p^{(\frac{(s^2+3s)}{2})r} - 2p^r$ and given that $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+3s)}{2})r} - 1$, we have that $\eta(\Gamma(R)) = p^{(\frac{(s^2+3s)}{2})r} - 1 - (p^{(\frac{(s^2+3s)}{2})r} - 2p^r) = p^{(\frac{(s^2+3s)}{2})r} - 1 - p^{(\frac{(s^2+3s)}{2})r} + 2p^r = 2p^r - 1$.

Case (ii): When $Char(R) = p^2,\ pu_i = 0$ we have,

$\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+3s+2)}{2})r} - 1$ and the rank of the adjacency matrix related to $\Gamma(R)$ is $p^{(\frac{(s^2+3s)}{2})r} - 1$. Therefore

$\eta(\Gamma(R)) = p^{(\frac{(s^2+3s+2)}{2})r} - 1 - (p^{(\frac{(s^2+3s)}{2})r} - 1) = p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}$, which is the nullity of $\Gamma(R)$ in this case.

Case (iii): When $Char(R) = p^2$ where $pu_i \neq 0$,

$\eta(\Gamma(R)) = p^{(\frac{(s^2+5s+2)}{2})r} - 1 - (p^{(\frac{(s^2+3s+2)}{2})r}) = p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 1$ since $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1$ and the rank of the adjacency matrix associated with $\Gamma(R)$ in this case is $p^{(\frac{(s^2+3s+2)}{2})r}$. This gives $\eta(\Gamma(R))$ as desired.

Case (iv): When $Char(R) = p^3$,

$\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+5s+4)}{2})r} - 1$ and the rank of the adjacency matrix is $p^{(\frac{(s^2+3s+2)}{2})r} + 2$.

We therefore obtain $\eta(\Gamma(R))$ as

$\eta(\Gamma(R)) = p^{(\frac{(s^2+5s+4)}{2})r} - 1 - (p^{(\frac{(s^2+3s+2)}{2})r} + 2) = p^{(\frac{(s^2+5s+4)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 3.$

□

Proposition 5.1.3. *Let $\Gamma(R)$ be the zero divisor graph of the classes of rings given by Constructions I, II and III in Sections 3.3.1, 3.4.1 and 3.5.1 respectively and $[L]$ be the Laplacian matrix associated with $\Gamma(R)$. Then for $r,\ s \in \mathbb{Z}^+,\ p$ prime integer and s fixed, we have $\eta(\Gamma(R)) = 1$.*

Proof. We show that the nullity, $\eta(\Gamma(R))$ of $\Gamma(R)$ with respect to the Laplacian matrix $[L]$ is 1 for all the characteristics $p^k, 1 \leq k \leq 3$.

When $Char(R) = p$ and $p \geq 2$.

It was established that $\mid \Gamma(R) \mid = p^{(\frac{(s^2+3s)}{2})r} - 1$ and the rank of the Laplacian matrix is $p^{(\frac{(s^2+3s)}{2})r} - 2$. We obtain the nullity of $\Gamma(R)$ with respect to $[L]$ as follows

$\eta(\Gamma(R)) = p^{(\frac{(s^2+3s)}{2})r} - 1 - (p^{(\frac{(s^2+3s)}{2})r} - 2) = 1.$

When $Char(R) = p^2$ and $pu_i = 0$,

$\mid \Gamma(R) \mid= p^{(\frac{(s^2+3s+2)}{2})r} - 1$ and the rank of Laplacian matrix is $p^{(\frac{(s^2+3s+2)}{2})r} - 2$ so we obtain $\eta(\Gamma(R)) = p^{(\frac{(s^2+3s+2)}{2})r} - 1 - (p^{(\frac{(s^2+3s+2)}{2})r} - 2) = 1$.

Further, for $Char(R) = p^2$ and $pu_i \neq 0$,

$\eta(\Gamma(R)) = p^{(\frac{(s^2+5s+2)}{2})r} - 1 - (p^{(\frac{(s^2+5s+2)}{2})r} - 2) = 1$ since $\mid \Gamma(R) \mid= p^{(\frac{(s^2+5s+2)}{2})r} - 1$ and $rank([L]) = p^{(\frac{(s^2+5s+2)}{2})r} - 2$.

The result is similar to the proof for $\eta(\Gamma(R))$ in the case when $Char(R) = p^3$. □

Proposition 5.1.4. *Let $\Gamma(R)$ be the zero divisor graph of classes of rings described by Constructions I, II, III, and IV in 4.2.1, 4.3.1, 4.4.1 and 4.5.1 respectively and $[A]_{p^{(h+(k-1))r}-1}$ be the adjacency matrix associated with $\Gamma(R)$. Then for $r \in \mathbb{Z}^+$, p prime and h, the dimension of $R'-$modules, we have*

$$\eta(\Gamma(R)) = \begin{cases} p^{(h-1)r} - 1, & \text{when } Char(R) = p; \\ p^{(h+2)r} - p^r - 1, & \text{when } Char(R) = p^2; \\ p^{(h+2)r} - p^r - 2, & \text{when } Char(R) = p^3; \\ p^{(h+2)r} - p^{hr} - p^{(h-2)r} - 2, & \text{when } Char(R) = p^4. \end{cases}$$

Proof. The proof is similar to that of Proposition 5.1.2. □

Definition 5.1.3. *[63] Let $G = (V, E)$ be the graph $\Gamma(R)$ and v_i be any vertex in $V(\Gamma(R))$. The subgraph obtained by removing $v_i \in V(\Gamma(R))$ is an induced subgraph and $V - v_i$ is called the removal of a vertex.*

Definition 5.1.4. *[63] Let $G = (V, E)$ be a graph and $e_i = (v_i, v_j)$ be any edge in $E(\Gamma(R))$, then the operation of removing e_i gives an induced subgraph with edge set $E - e_i$ and is called the removal of an edge.*

Proposition 5.1.5. *Let $\Gamma_v(R)$ be the induced subgraph of the zero divisor graph of classes of rings given by Constructions I, II, III, and IV in 4.2.1, 4.3.1, 4.4.1 and 4.5.1 respectively obtained by removal of $v_i \in \Gamma(R)$ of maximum degree $p^{(h+(k-1))r} - 2$. If $[A^v]$ is the adjacency matrix of $\Gamma_v(R)$ then for any $r, k \in \mathbb{Z}^+$, prime integer p and h, the dimension of $R'-$modules, we have*

(i) The order of $[A^v]$ is $(p^{h+(k-1))r} - 2) \times (p^{h+(k-1))r} - 2)$,

(ii) $rank([A^v]) = p^{(h+(k-2))r} - 1$,

(iii) $Det([A^v]) = Det([A])$, *and*

(iv) $\sigma_{point}((\Gamma_v(R)) \leq \sigma_{point}(\Gamma(R))$.

Proof. (i) Since $\Gamma_v(R)$ is obtained by the operation $V - v_i$ from $\Gamma(R)$, $\mid V(\Gamma(R)) \mid - \mid V(\Gamma_v(R)) \mid = 1$. Since $\mid V(\Gamma(R)) \mid = p^{(h+(k-1))r} - 1$ equivalent to the number of rows and columns of $[A]_{p^{(h+(k-1))r}-1}$, and $[A^v]$ is obtained from $V(\Gamma(R)) - v_i$ implies $\Gamma_v(R)$, we have that the dimension of $[A^v] = (p^{(h+(k-1))r} - 1) - 1 = p^{(h+(k-1))r} - 2$. The matrix $[A^v]$ being square, the order follows.
ii) Given that the rank of $[A]_{p^{(h+(k-1))r}-1}$ is of the form $p^{(h+(k-1))r}$ and any sub matrix $[A^v]$ of $[A]_{p^{(h+(k-1))r}-1}$ obtained by removal of a vertex $v_i \in V(\Gamma(R))$ is of order $p^{(h+(k-1))r} - 2$, we perform elementary row operation on $[A^v]$ to obtain $p^{(h+(k-2))r} - 1$ linearly independent vectors $x_1, x_2, \cdots, x_{(h+(k-2))r-1}$ which span the whole matrix space due to the removal of a vertex v_i from $V(\Gamma(R))$. The operation $V - v_i$ leads to a deletion of one row and one column from $[A]_{p^{(h+(k-1))r}-1}$ hence the result.
(iii) The proof to show that $[A^v]$ is singular follows from the fact that $[A^v]$ inherits properties from $[A]_{p^{(h+(k-1)r}}$, and that $[A]_{p^{(h+(k-1)r}}$ is singular, follows from the fact that $[A^v]$ is singular.
(iv) Consider the subgraph $\Gamma_v(R)$ obtained by removing the vertex v_i of maximum degree. Let $x = (x_1, x_2, \cdots, x_{p^{(h+(k-1))r}-2})^T$ be the eigenvectors corresponding to $\sigma_{point}(\Gamma_v(R))$. Let $\overline{x} = (x_\circ, \cdots, x_{p^{(h+(k-1))r}-2})^T$ be the eigenvectors corresponding to the deleted vertex from $\Gamma(R)$ to obtain $\Gamma_v(R)$. Set $x_\circ = 0$, we have $Row(A(\Gamma(R), \overline{x}) = Row(A^v(\Gamma_v(R)), x)$. Therefore,
$\sigma_{point}(\Gamma(R)) \geq Row(A(\Gamma(R), \overline{x}) = Row(A^v(\Gamma_v(R)), x) = \sigma_{point}(\Gamma_v(R))$ which implies that $\sigma_{point}(\Gamma(R)) \geq \sigma_{point}(\Gamma_v(R))$.

□

Proposition 5.1.6. *Let* $\Gamma_v(R)$ *be the induced subgraph of the zero divisor graph of classes of rings described by Constructions I, II, III and IV in 4.2.1, 4.3.1, 4.4.1 and 4.5.1 respectively obtained by removal of* $v_i \in \Gamma(R)$ *of maximum degree* $p^{(h+(k-1))r} - 2$.

If $[L^v]$ *is the Laplacian matrix of* $\Gamma_v(R)$ *then for any* $r, k \in \mathbb{Z}^+$, *a prime integer* p, *and* h *the dimension of* $R'-$*modules; then*

(i) The order of $[L^v]$ *is* $(p^{h+(k-1))r} - 2) \times (p^{h+(k-1))r} - 2)$,

(ii) $rank([L^v]) = rank([L]_{p^{(h+(k-1))r}-1}) = p^{(h+(k-2))r} + 1$, *and*

(iii) $Det([L^v]) = Det([L])$.

Proof. The proof is similar to the ones in Proposition 5.1.5. □

We now look aat properties of matrices obtained by raising them to finite powers in the following results.

Proposition 5.1.7. *Let* $\Gamma(R)$ *be the zero divisor graph of the classes of rings described by the Constructions I, II, III and IV in Subsections 4.2.1, 4.3.1, 4.4.1 and 4.5.2 respectively. If the diameter of* $\Gamma(R)$ *is* q *and* $[A]_{p^{(h+(k-1))r}}$ *is the adjacency matrix of* $\Gamma(R)$, *then for any prime integer* p *and* $r, k \in \mathbb{Z}^+$ *with* h *the dimension of* $R'-$*modules, then the matrix given by* $([A]_{p^{(h+(k-1))r}-1})^q + ([A]_{p^{(h+(k-1))r}-1})^{q-1}$ *has nonzero entries.*

Proof. If q is even, let $v_i, v_j \in V(\Gamma(R))$ and $d_{i,j}$ be the distance from vertex v_i to v_j. Then we have a walk of length $d_{i,j} + q - d_{i,j} = q$ between the vertices v_i and v_j. Therefore, the entry $a_{ij} \in [A]_{p^{(h+(k-1))r}-1} > 0$. If there is an odd $d_{i,j}$, then $q - d_{i,j} - 1$ is even thus we have a v_i, v_j-walk of length $d_{i,j} + (q - d_{i,j} - 1) = q - 1$ resulting to a_{ij}^{th}-entry of the matrix $([A]_{p^{(h+(k-1))r}-1})^{q-1}$ is greater than 0. Since $a_{ij} \in ([A]_{p^{(h+(k-1))r}-1})^q$ and $([A]_{p^{(h+(k-1))r}-1})^{q-1}$ are non-negative, it implies that $([A]_{p^{(h+(k-1))r}-1})^q + ([A]_{p^{(h+(k-1))r}-1})^{q-1}$ is of non-zero entries. □

Lemma 5.1.1. *Suppose* q *is the smallest positive integer such that* $([A]_{p^{(h+(k-1))r}-1} + I)^q$ *is of non-zero entries then,* $q = diam(\Gamma(R))$.

Proof. Let $([A]_{p^{(h+(k-1))r}-1})^q = I + q([A]_{p^{(h+(k-1))r}-1}) + \cdots + ([A]_{p^{(h+(k-1))r}-1} + I)^q$ with a_{ij} being non-negative. Therefore $([A]_{p^{(h+(k-1))r}-1})^q$ has a non-zero $a_{ij}^{th}-$ entry only if a_{ij}^k is greater than zero for $1 \leq k \leq q$. Assume $([A]_{p^{(h+(k-1))r}-1})^q + ([A]_{p^{(h+(k-1))r}-1})^{q-1}$

has a_{ij} zero entry, then there exist $([A]_{p^{(h+(k-1))r}-1})^k$ with $1 \leq k \leq q-2$ such that a_{ij}^{th}-entry is non-zero.

Thus we have walks of length $k+(q-k)=q$ or $k+(q-k-1)=q-1$ between the vertices v_i, v_j. This implies that a_{ij}^{th} entry of the sum $([A]_{p^{(h+(k-1))r}-1})^{q-1} + ([A]_{p^{(h+(k-1))r}-1})^q$ is not zero, a contradiction. Therefore the smallest natural number such that $([A]_{p^{(h+(k-1))r}-1})^q + ([A]_{p^{(h+(k-1))r}-1})^{q-1}$ has no zero entries is q. Hence follows the result from the previous Proposition 5.1.7 □

Proposition 5.1.8. *Let $\Gamma(R)$ be the zero divisor graph of the classes of rings given by Constructions I, II, III and IV in Subsections 4.2.1, 4.3.1, 4.4.1 and 4.5.1 respectively such that $\mid V(\Gamma(R)) \mid = p^{(h+(k-1))r}-1$ for any p prime, $k, r \in \mathbb{Z}^+$ and h the dimension of $R'-$modules. If $[A]$ is the adjacency matrix of $\Gamma(R)$, then the entry a_{ij}^f of $[A]^f$ is the number of $v_i - v_j$ distinct walks of length f in $\Gamma(R)$ for any $f = 1, 2, \cdots$ and $i, j = 1, 2, \cdots, p^{(h+(k-1))}-1$.*

Proof. Proceeding by induction on f, let $f=1$. Then $[A]=[A]^1$ whose entries are

$$a_{ij} = \begin{cases} 1, & \text{if } v_i, v_j \text{ are adjacent;} \\ 0, & \text{if } v_i, v_j \text{ are non adjacent.} \end{cases}$$

Clearly, for all $v_i, v_j \in V(\Gamma(R))$, the walk $v_i - v_j$ is of length 1.

Assume the $v_i - v_j$ walks of length f in $\Gamma(R)$ is a_{ij}^f for any positive integer f. We illustrate that $[A]^{f+1}$ has an entry a_{ij}^{f+1} which gives $v_i - v_j$ walks of $f+1$ in length. This entry is represented by

$$[A]^{f+1} = \begin{bmatrix} a_{11}^{f+1} & a_{12}^{f+1} & \cdot\ \cdot\ \cdot & a_{1(p^{(h+(k-1))r}-1)}^{f+1} \\ a_{21}^{f+1} & a_{22}^{f+1} & \cdot\ \cdot\ \cdot & a_{1(p^{(h+(k-1))r}-1)}^{f+1} \\ \cdot & & & \\ \cdot & & a_{ij}^{f+1} & \\ \cdot & & & \\ a_{(p^{(h+(k-1))r}-1)1}^{f+1} & a_{(p^{(h+(k-1))r}-1)2}^{f+1} & \cdot\ \cdot\ \cdot & a_{(p^{(h+(k-1))r}-1)}^{f+1} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}^{f} & a_{12}^{f} & \cdot\ \cdot\ \cdot & a_{1(p^{(h+(k-1))r}-1)}^{f} \\ a_{21}^{f} & a_{22}^{f} & \cdot\ \cdot\ \cdot & a_{1(p^{(h+(k-1))r}-1)}^{f} \\ \cdot & & & \\ \cdot & & a_{ij}^{f} & \\ \cdot & & & \\ a_{(p^{(h+(k-1))r}-1)1}^{f} & a_{(p^{(h+(k-1))r}-1)2}^{f} & \cdot\ \cdot\ \cdot & a_{(p^{(h+(k-1))r}-1)}^{f} \end{bmatrix} \times$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1(p^{(h+(k-1))r}-1)} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{1(p^{(h+(k-1))r}-1)} \\ \cdot & & & & & \\ \cdot & & a_{mj} & & & \\ \cdot & & & & & \\ a_{(p^{(h+(k-1))r}-1)1} & a_{(p^{(h+(k-1))r}-1)2} & \cdot & \cdot & \cdot & a_{(p^{(h+(k-1))r}-1)} \end{bmatrix}.$$

Therefore, the a_{ij}^{f+1} entry can be obtained through the product of i^{th} row of $[A]^f$ and j^{th} column of $[A]$. That is,

$$a_{ij}^{f+1} = \sum_{m=1}^{p^{(h+(k-1))r}-1} v_m = a_{i1}^f a_{1j} + a_{i2}^f a_{2j} + \cdots + a_{i(p^{(h+(k-1))r}-1)}^f a_{(p^{(h+(k-1))r}-1)j}.$$

Thus each $v_i - v_j$ walk of length $f+1$ has length f of $v_i - v_m$ walks. v_m being adjacent to v_j implies that $v_i - v_j$ walks of length $f+1$ in $\Gamma(R)$ is a_{ij}^{f+1}. □

5.2 The Binding Number of $\Gamma(R)$ of the Classes of 3-Radical Zero and 4-Radical Zero Completely Primary Finite Rings

5.2.1 Introduction

The reason for rapid and steady development in the study of the binding number $b(\Gamma(R))$ of graphs is related to its diversity in both theoretical and real world applications in graph networks and communication links. This is an important characteristic quantity which is applicable in understanding the graph characteristics and vulnerability.

Definition 5.2.1. *The binding number of* $\Gamma(R)$ *denoted by* $b(\Gamma(R)) = \frac{|N(S)|}{|S|}$ *where* $S \subseteq V(\Gamma(R)), N(S) \neq V(\Gamma(R)), S \neq \Phi$ *such that*

(i) $N(S) \cup S = V(\Gamma(R))$,

(ii) $N(S) \cap S = \Phi$,

(iii) $deg(u) \leq deg(v)$ *for all* $u \in S,\ v \in N(S)$, *and*

(iv) no pair of vertices in S *are adjacent.*

The following are some results we have obtained on the binding number indices of the zero divisor graphs of the classes of 3-radical zero and 4-radical zero completely primary finite rings.

Proposition 5.2.1. *Let* $\Gamma(R)$ *be the zero divisor graph of a ring given by Construction I in Section 3.3. Then the binding number,*

$$b(\Gamma(R)) = \frac{p^{(\frac{(s^2+3s)}{2}-1)r}-1}{p^{(\frac{(s^2+3s)}{2})r}-p^{(\frac{(s^2+3s)}{2}-1)r}}.$$

Proof. Due to the fact that $N(S) = Ann(Z(R)) \setminus \{0\}$ and $S = V(\Gamma(R)) \setminus N(S)$. we have $\mid N(S) \mid = p^{(\frac{(s^2+3s)}{2}-1)r} - 1$.
But $\mid S \mid = \mid V(\Gamma(R)) \setminus N(S) \mid = p^{(\frac{(s^2+3s)}{2})r} - 1 - (p^{(\frac{(s^2+3s)}{2}-1)r} - 1) = p^{(\frac{(s^2+3s)}{2})r} - 1 - p^{(\frac{(s^2+3s)}{2}-1)r} + 1 = p^{(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s)}{2}-1)r}$. Therefore, the binding number of $\Gamma(R)$,

$$b(\Gamma(R)) = \frac{\mid N(S) \mid}{\mid S \mid} = \frac{p^{(\frac{(s^2+3s)}{2}-1)r}-1}{p^{(\frac{(s^2+3s)}{2})r}-p^{(\frac{(s^2+3s)}{2}-1)r}}.$$

□

Proposition 5.2.2. *Let* $\Gamma(R)$ *be the zero divisor graph of a ring given by Construction II in Section 3.4. Then the binding number,*

$$b(\Gamma(R)) = \begin{cases} \frac{p^{(\frac{(s^2+3s)}{2})r}-1}{p^{(\frac{(s^2+3s+2)}{2})r}-p^{(\frac{(s^2+3s)}{2})r}}, & pu_i = 0; \\ \frac{p^{(\frac{(s^2+3s+2)}{2})r}-1}{p^{(\frac{(s^2+5s+2)}{2})r}-p^{(\frac{(s^2+3s+2)}{2})r}}, & pu_i \neq 0. \end{cases}$$

Proof. Case (i): $pu_i = 0$.
Since $N(S) = Ann(Z(R)) \setminus \{0\}$ and $S = V(\Gamma(R)) \setminus N(S)$,
it is established that $\mid N(S) \mid = p^{(\frac{(s^2+3s)}{2})r} - 1$.
But $\mid S \mid = \mid V(\Gamma(R)) \setminus N(S) \mid = p^{(\frac{(s^2+3s+2)}{2})r} - 1 - (p^{(\frac{(s^2+3s)}{2})r} - 1) = p^{(\frac{(s^2+3s+2)}{2})r} - 1 - p^{(\frac{(s^2+3s)}{2})r} + 1 = p^{(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+3s)}{2})r}$. Therefore, the binding number of $\Gamma(R)$,

$$b(\Gamma(R)) = \frac{\mid N(S) \mid}{\mid S \mid} = \frac{p^{(\frac{(s^2+3s)}{2})r}-1}{p^{(\frac{(s^2+3s+2)}{2})r}-p^{(\frac{(s^2+3s)}{2})r}}.$$

Case (ii): $pu_i \neq 0$.
Given that $\mid V(\Gamma(R)) \mid = p^{(\frac{(s^2+5s+2)}{2})r} - 1$, consider the $Ann(Z(R))^* = N(S)$. We notice

that $N(S) = p^{(\frac{(s^2+3s+2)}{2})r} - 1$ and therefore

$$| S |= (p^{(\frac{(s^2+5s+2)}{2})r} - 1) - (p^{(\frac{(s^2+3s+2)}{2})r} - 1) = p^{(\frac{(s^2+5s+2)}{2})r} - 1 - p^{(\frac{(s^2+3s+2)}{2})r} + 1 =$$
$$p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}.$$

So,

$$b(\Gamma(R)) = \frac{p^{(\frac{(s^2+3s+2)}{2})r} - 1}{p^{(\frac{(s^2+5s+2)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}}.$$

□

Proposition 5.2.3. *Let R be a ring described by Construction III in Section 3.5 and $\Gamma(R)$ be its zero divisor graph. Then,*

$$b(\Gamma(R)) = \frac{p^{(\frac{(s^2+3s+2)}{2})r} - 1}{p^{(\frac{(s^2+5s+4)}{2})r} - (p^{(\frac{(s^2+3s+2)}{2})r})}.$$

Proof. The proof follows from the fact that

$$| N(S) |=| Ann(Z(R))^* |= p^{(\frac{(s^2+3s+2)}{2})r} - 1$$

for the rings of characteristic p^3 and $S = V(\Gamma(R)) \setminus N(S)$. We have that $| S |= (p^{(\frac{(s^2+5s+4)}{2})r} - 1) - (p^{(\frac{(s^2+3s+2)}{2})r} - 1) = p^{(\frac{(s^2+5s+4)}{2})r} - 1 - p^{(\frac{(s^2+3s+2)}{2})r} + 1 = p^{(\frac{(s^2+5s+4)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}$

Using the ratio $\frac{|N(S)|}{|S|}$, we obtain

$$b(\Gamma(R)) = \frac{p^{(\frac{(s^2+3s+2)}{2})r} - 1}{p^{(\frac{(s^2+5s+4)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r}}.$$

□

Proposition 5.2.4. *Let R be rings given by Constructions I, II, III and IV in Sections 4.2, 4.3, 4.4 and 4.5 respectively and $\Gamma(R)$ be the zero divisor graph. Then the binding number,*

$$b(\Gamma(R)) = \begin{cases} \frac{1}{p^{hr}-2}, & \text{when } Char(R) = p; \\ \frac{p^{hr}-1}{p^{(h+3)r}-p^{hr}}, & \text{when } Char(R) = p^2; \\ \frac{p^{(h-1)r}}{p^{(h+3)r}-p^{(h-1)r}}, & \text{when } Char(R) = p^3 \text{ and } p^4. \end{cases}$$

Proof. With the fact that $N(S) = Ann(Z(R))^*$ and $S = V(\Gamma(R)) \setminus N(S)$, applying the ratio $\frac{|N(S)|}{|S|}$, the results follows from the proofs in Propositions 5.2.1, 5.2.2 and 5.2.3. □

5.3 The Wiener Index of $\Gamma(R)$ and its Invariants for the Classes of 3-Radical Zero and 4-Radical Zero Completely Primary Finite Rings

5.3.1 Introduction

Other than the binding number, we also present some findings on the Wiener index $W(\Gamma(R))$, the average disorder number $A(\Gamma(R))$ and the average distance index $\mu(\Gamma(R))$ of the zero divisor graph $\Gamma(R)$. The Wiener index denoted as W and also known as path number or the Wiener number is a graph index defined on a graph by n nodes as

$$W = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[d]_{ij}$$

where $[d]_{ij}$ is the graph distance matrix. The Wiener index $W(\Gamma(R))$ of the graph G with vertex count $\mid V(\Gamma(R)) \mid$ has a relationship with the average disorder number of the zero divisor graph $A(\Gamma(R)) = \frac{2W(\Gamma(R))}{|V(\Gamma(R))|}$ and the average distance $\mu(\Gamma(R))$ between the vertices of $\Gamma(R)$ which is given by

$$\mu(\Gamma(R)) = \frac{W(\Gamma(R))}{\binom{\mid V(\Gamma(R)) \mid}{2}}.$$

In chemical graph theory, computations for Wiener indices for cyclic carbon-chained organic compounds and its applications cannot be underestimated. This index has been quite handy in determination of the boiling points and polarity number of alkanes and their branched isomers. Further, the most common natural field in the application of the Wiener index is the quantitative structure relationships especially in the estimation of emission spectra of the ultra violet radiations of α and β-unsaturated ketones. We therefore present the following results on the Wiener index of $\Gamma(R)$ and other results describing average disorder number and the average distance indices of $\Gamma(R)$ due to their close interdependence with the Wiener index.

Proposition 5.3.1. *Let $\Gamma(R)$ be the zero divisor graph of the classes of rings given by Constructions I, II and III in Sections 3.3, 3.4 and 3.5 respectively. Then for any prime integer p and $r, s \in \mathbb{Z}^+$ with s fixed, the Wiener index, $W(\Gamma(R))$*

$$= \begin{cases} \frac{1}{2}(2p^{(\frac{2(s^2+3s)}{2}-1)r} + p^{2(\frac{(s^2+3s)}{2}-1)r} - p^{(\frac{(s^2+3s)}{2})r} - 5p^{(\frac{(s^2+3s)}{2}-1)r} + 2), & \text{if } char(R) = p; \\ \frac{1}{2}(2p^{(\frac{2(s^2+3s)}{2}+2)r} + p^{2(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 5p^{(\frac{(s^2+3s)}{2})r} + 2), & \text{if } char(R) = p^2, pu_i = 0; \\ \frac{1}{2}(2p^{(\frac{(2s^2+8s+4)}{2})r} + p^{2(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} - 5p^{(\frac{(s^2+3s+2)}{2})r} + 2), & \text{if } char(R) = p^2, pu_i \neq 0; \\ \frac{1}{2}(2p^{(\frac{(2s^2+10s+6)}{2})r} + p^{2(\frac{(2s^2+10s+4)}{2})r} - p^{2(\frac{(s^2+5s+4)}{2})r} - 5p^{(\frac{(s^2+5s+2)}{2})r} + 2), & \text{if } char(R) = p^3. \end{cases}$$

Proof. Case (i): $Char(R) = p$. The maximum degree of $v_i \in V(\Gamma(R)) =$ $p^{(\frac{(s^2+3s)}{2})r} - 2$ and there are $p^{(\frac{(s^2+3s)}{2}-1)r} - 1$ of such vertices. This is due to the fact that $\mid ann(Z(R))^* \mid = p^{(\frac{(s^2+3s)}{2}-1)r} - 1$. Therefore, the sum of minimum distances between v_i of a maximum degree and any other vertex $v_j \in V(\Gamma(R))$ is $(p^{(\frac{(s^2+3s)}{2})r} - 2)(p^{(\frac{(s^2+3s)}{2}-1)r} - 1)$. For the vertices of minimum degrees, each is of degree $p^{(\frac{(s^2+3s)}{2}-1)r} - 1$ and sum of the distances between a vertex of minimum degree and any other vertex in the set of vertices of minimum degree is $p^{(\frac{(s^2+3s)}{2})r} - 2$. Since they are $p^{(\frac{(s^2+3s)}{2}-1)r}$ in number, then from the argument we obtain the sum as $((p^{(\frac{(s^2+3s)}{2}-1)r} - 1) + (p^{(\frac{(s^2+3s)}{2})r} - 2))p^{(\frac{(s^2+3s)}{2}-1)r}$. The multiple $\frac{1}{2}$ handles the fact that each path between v_i and v_j is also counted as the path between v_j and v_i hence $W(\Gamma(R)) = \frac{1}{2}(p^{(\frac{(s^2+3s)}{2})r} - 2)(p^{(\frac{(s^2+3s)}{2}-1)r} - 1) + ((p^{(\frac{(s^2+3s)}{2}-1)r} - 1) + (p^{(\frac{(s^2+3s)}{2})r} - 2))p^{(\frac{(s^2+3s)}{2}-1)r} =$
$\frac{1}{2}(p^{(\frac{2(s^2+3s)}{2}-1)r} - p^{(\frac{(s^2+3s)}{2})r} - 2p^{(\frac{(s^2+3s)}{2}-1)r} + 2 + p^{2(\frac{(s^2+3s)}{2}-1)r} + p^{(\frac{2(s^2+3s)}{2}-1)r} - 3p^{(\frac{(s^2+3s)}{2}-1)r}) =$

$$\frac{1}{2}((2p^{(\frac{2(s^2+3s)}{2}-1)r} + p^{2(\frac{(s^2+3s)}{2}-1)r} - 5p^{(\frac{(s^2+3s)}{2}-1)r} - p^{(\frac{(s^2+3s)}{2})r} + 2)).$$

Case (ii): $Char(R) = p^2$ where $pu_i = 0$.

Using the same argument similar to $char(R) = p$, we obtain
$W(\Gamma(R)) = \frac{1}{2}((p^{(\frac{(s^2+3s+2)}{2})r} - 2)(p^{(\frac{(s^2+3s)}{2})r} - 1) + ((p^{(\frac{(s^2+3s)}{2})r} - 1) + (p^{(\frac{(s^2+3s+2)}{2})r})p^{(\frac{(s^2+3s)}{2})r}) =$

$\frac{1}{2}(p^{(\frac{2(s^2+3s)}{2}+2)r} - p^{(\frac{(s^2+3s+2)}{2})r} - 2p^{(\frac{(s^2+3s)}{2})r} + 2 + p^{(\frac{2(s^2+3s)}{2})r} + p^{(\frac{2(s^2+3s)}{2}+2)r} - 3p^{(\frac{(s^2+3s)}{2})r}) =$

$$\frac{1}{2}(2p^{(\frac{2(s^2+3s)}{2}+2)r} + p^{2(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 5p^{(\frac{(s^2+3s)}{2})r} + 2).$$

Case (iii): $Char(R) = p^2$ where $pu_i \neq 0$,
$W(\Gamma(R)) = \frac{1}{2}((p^{(\frac{(s^2+5s+2)r}{2})r}) - 2)(p^{(\frac{(s^2+3s+2)}{2})r} - 1) + ((p^{(\frac{(s^2+3s+2)}{2})r} - 1) + (p^{(\frac{(s^2+5s+2)}{2})r} -$

$2))p^{(\frac{(s^2+3s+2)}{2})r}) =$

$\frac{1}{2}(p^{(\frac{(2s^2+8s+4)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+3s+2)}{2})r} + 2 + (p^{2(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(2s^2+8s+4)}{2})r} - 3p^{(\frac{(s^2+3s+2)}{2})r}) =$

$$\frac{1}{2}(2p^{(\frac{(s^2+8s+4)}{2})r} + p^{2(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} - 5p^{(\frac{(s^2+3s+2)}{2})r} + 2).$$

Case (iv): $Char(R) = p^3$.

We have

$W(\Gamma(R)) = \frac{1}{2}((p^{(\frac{(s^2+5s+4)}{2})r} - 2)(p^{(\frac{(2s^2+5s+2)}{2})r} - 1) + ((p^{(\frac{(s^2+5s+2)}{2})r} - 1) + (p^{(\frac{(s^2+5s+4)}{2})r}) - 2)p^{(\frac{(s^2+5s+2)}{2})r}) = \frac{1}{2}(2p^{(\frac{(2s^2+10s+6)}{2})r} - p^{(\frac{(s^2+5s+4)}{2})r} - 2p^{(\frac{(s^2+5s+2)}{2})r} + 2 + (p^{(\frac{(2s^2+10s+4)}{2})r} + p^{(\frac{(2s^2+10s+6)}{2})r} - 3p^{(\frac{(s^2+5s+2)}{2})r}) =$

$$\frac{1}{2}(2p^{(\frac{(2s^2+10s+6)}{2})r} + p^{(\frac{(2s^2+10s+4)}{2})r} - p^{(\frac{(s^2+5s+4)}{2})r} - 5p^{(\frac{(s^2+5s+2)}{2})r} + 2).$$

□

Proposition 5.3.2. *Let $\Gamma(R)$ be the zero divisor graph of the classes of rings given by Constructions I, II and III in Sections 3.3, 3.4 and 3.5 respectively and $W(\Gamma(R))$ be its Wiener index. Then for any prime integer p, positive integers r, s with s fixed, the average distance of $\Gamma(R)$*

$$\mu(\Gamma(R)) = \begin{cases} \dfrac{\frac{1}{2}(2p^{(\frac{(2(s^2+3s)}{2}-1)r} + p^{2(\frac{(s^2+3s)}{2}-1)r} - p^{(\frac{(s^2+3s)}{2})r} - 5p^{(\frac{(s^2+3s)}{2}-1)r} + 2)}{(p^{(\frac{(s^2+3s)}{2})r} - 1)(p^{(\frac{(s^2+3s)}{2})r} - 2)}, & Char(R) = p; \\ \dfrac{\frac{1}{2}(2p^{(\frac{2(s^2+3s)}{2}+2)r} + p^{(\frac{2(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 5p^{(\frac{(s^2+3s)}{2})r} + 2)}{(p^{(\frac{(s^2+3s+2)}{2})r} - 1)(p^{(\frac{(s^2+3s+2)}{2})r} - 2)}, & Char(R) = p^2, pu_i = 0; \\ \dfrac{\frac{1}{2}(2p^{(\frac{(2s^2+8s+4)}{2})r} + p^{2(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} - 5p^{(\frac{(s^2+3s+2)}{2})r} + 2)}{(p^{(\frac{(s^2+5s+2)}{2})r} - 1)(p^{(\frac{(s^2+5s+2)}{2})r} - 2)}, & Char(R) = p^2, pu_i \neq 0; \\ \dfrac{\frac{1}{2}(2p^{(\frac{(2s^2+10s+6)}{2})r} + p^{2(\frac{(2s^2+10s+4)}{2})r} - p^{2(\frac{(s^2+5s+4)}{2})r} - 5p^{(\frac{(s^2+5s+2)}{2})r} + 2)}{(p^{(\frac{(s^2+5s+4)}{2})r} - 1)(p^{(\frac{(s^2+5s+4)}{2})r} - 2)}, & Char(R) = p^3. \end{cases}$$

Proof. The proof follows from the previous Proposition 5.3.1 and the fact that $\mu(\Gamma(R)) = \frac{W(\Gamma(R))}{\binom{\mid V(\Gamma(R)) \mid}{2}}$. □

Proposition 5.3.3. *Let $\Gamma(R)$ be the zero divisor graph of the classes of rings described by Constructions I, II and III in Sections 3.3, 3.4 and 3.5 respectively. Then for any prime integer p, positive integers r, s and s fixed, the average disorder number of the* *zero divisor graph*

$$A(\Gamma(R)) = \begin{cases} \frac{2p^{(\frac{(2(s^2+3s)}{2}-1)r}+p^{2(\frac{(s^2+3s)}{2}-1)r}-p^{(\frac{(s^2+3s)}{2})r}-5p^{(\frac{(s^2+3s)}{2}-1)r}+2}{(p^{(\frac{((s^2+3s)}{2})r}-1)}, & Char(R) = p; \\ \frac{2p^{(\frac{2(s^2+3s)}{2}+2)r}+p^{(\frac{2(s^2+3s)}{2})r}-p^{(\frac{(s^2+3s+2)}{2})r}-5p^{(\frac{(s^2+3s)}{2})r}+2}{(p^{(\frac{((s^2+3s+2)}{2})r}-1)}, & Char(R) = p^2, pu_i = 0; \\ \frac{2p^{(\frac{(2s^2+8s+4)}{2})r}+p^{2(\frac{(s^2+3s+2)}{2})r}-p^{(\frac{(s^2+5s+2)}{2})r}-5p^{(\frac{(s^2+3s+2)}{2})r}+2}{(p^{(\frac{((s^2+5s+2)}{2})r}-1)}, & Char(R) = p^2, pu_i \neq 0; \\ \frac{2p^{(\frac{(2s^2+10s+6)}{2})r}+p^{2(\frac{(2s^2+10s+4)}{2})r}-p^{2(\frac{(s^2+5s+4)}{2})r}-5p^{(\frac{(s^2+5s+2)}{2})r}+2}{(p^{(\frac{((s^2+5s+4)}{2})r}-1)}, & Char(R) = p^3. \end{cases}$$

Proof. Using the ratio $A(\Gamma(R)) = \frac{2W(\Gamma(R))}{|V(\Gamma(R))|}$ and the Wiener indices obtained, the result follows. □

Proposition 5.3.4. *Let $\Gamma(R)$ be the zero divisor graph of a ring given by Construction I in Section 4.2. Then the Wiener index,*

$$W(\Gamma(R)) = \frac{1}{2}(p^{(h+1)r} + p^{2hr} - 2(p^{2r} + p^r) + 2).$$

Proof. Let V_1, V_2 and V_3 be the order of partitioning of the vertices in $\Gamma(R)$. Consider $V_1 = Ann(Z(R)) \setminus \{0\}$. As in Proposition 4.2.1(ii), $\mid V_1 \mid = p^r - 1$ and the degree of each vertex $x \in V_1, deg(x \in V_1) = p^{hr} - 1 - 1 = p^{hr} - 2$ due to avoidance of self annihilation. Therefore the sum of the distances between vertices $x \in V_1$ and any other vertex in $V(\Gamma(R))$ is $(p^{hr} - 2)(p^r - 1) = p^{(h+1)r} - p^{hr} - 2p^r + 2$.

Next, consider the set of vertices V_2 which are linked by edges with vertices in V_1 and among themselves such that $d(x, y \in V_2) = p^{hr} + 2$. Since $\mid V_2 \mid = p^r$, we have that

$$\sum_{y \in V_2} d(x, y) = p^r(p^{hr} + 2) = p^{(h+1)r} + 2p^r.$$

Finally, let V_3 be the set of vertices such that $z \in V_3$ is only adjacent $x \in V_1$. Therefore, $\mid V_3 \mid = (p^{hr} - 1) - (p^r + p^r - 1) = p^{hr} - 2p^r$ and the distance between every vertex in V_3 and all other vertices in $\Gamma(R)$ is $p^{hr} + p^r + 1$. Therefore, the sum of the minimum distances between $z \in V_3$ and vertices $V(\Gamma(R))$ is obtained as

$$\sum_{z \in V_3} d(x, z) = (p^{hr} - 2p^r)(p^{hr} + p^r + 1) =$$

$$p^{2hr} + p^{(h+1)r} + p^{hr} - 2p^{(h+1)r} - 2p^{2r} - 2p^r = p^{2hr} - p^{(h+1)r} + p^{hr} - 2p^{2r} - 2p^r.$$

Therefore,

$$W(\Gamma(R)) = \frac{1}{2}(p^{(h+1)r} - p^{hr} - 2p^r + 2 + p^{(h+1)r} + 2p^r + p^{2hr} - p^{(h+1)r} + p^{hr} - 2p^{2r} - 2p^r)$$

which simplifies to

$$\frac{1}{2}(p^{(h+1)r} + p^{2hr} - 2p^{2r} - 2p^r + 2) = \frac{1}{2}(p^{(h+1)r} + p^{2hr} - 2(p^{2r} + p^r) + 2).$$

□

Proposition 5.3.5. *Let $W(\Gamma(R))$ be the Wiener index of the zero divisor graph of a ring given by Construction I in Section 4.2. Then the average disorder number $A(\Gamma(R))$ and the average distance index $\mu(\Gamma(R))$ are given by;*

(i) $A(\Gamma(R)) = \frac{p^{(h+1)r}+p^{2hr}-2(p^{2r}+p^r)+2}{p^{hr}-1}$.

(ii) $\mu(\Gamma(R)) = \frac{\frac{1}{2}(p^{(h+1)r}+p^{2hr}-2(p^{2r}+p^r)+2)}{(p^{hr}-1)(p^{hr}-2)}$.

Proof. From the fact that $\mid V(\Gamma(R)) \mid = p^{hr} - 1$, using the ratios $A(\Gamma(R)) = \frac{2W(\Gamma(R))}{|V(\Gamma(R))|}$ and the average distance $\mu(\Gamma(R))$ between the vertices of $\Gamma(R)$, $\mu(\Gamma(R)) = \frac{W(\Gamma(R))}{\binom{\mid V(\Gamma(R)) \mid}{2}}$, the results follows from the previous Proposition 5.3.4. □

Proposition 5.3.6. *Let $\Gamma(R)$ be the zero divisor graph of classes of rings with characteristics p^2, p^3 and p^4 respectively given by the Construction II and $W(\Gamma(R))$ be the Wiener index. Then,*

$$W(\Gamma(R)) \begin{cases} \frac{1}{2}(p^{(h+2)r} + p^{(2h+1)r} - 2(p^{3r} + p^{2r})), & \textit{when } Char(R) = p^2; \\ \frac{1}{2}(p^{(h+3)r} + p^{(2h+2)r} - 2(p^{4r} + p^{3r})), & \textit{when } Char(R) = p^3; \\ \frac{1}{2}(p^{(h+4)r} + p^{(2h+3)r} - 2(p^{5r} + p^{4r})), & \textit{when } Char(R) = p^4. \end{cases}$$

Proof. The proof follows a similar trend as in the proof of Proposition 5.3.4. □

5.4 The Bounds On Zagreb Indices of the Zero Divisor Graphs $\Gamma(R)$ of Classes of 4-Radical Zero Completely Primary Finite Rings

5.4.1 Introduction

Let $G = \Gamma(R)$ be a simple graph such that $G = (V, E)$ whose vertex set $V(G)$ consists of elements $\{v_1, \cdots, v_n\}$ such that $| V(G) |= n$ and the set of edges $E(G)$ of order m. Given that the minimum degree of G is denoted by $\delta(G)$ and $\Delta(G)$ is the maximum degree. Let $d_i = deg_{\Gamma(R)}(v_i),\ i = 1, 2, \cdots, n$ be the vertex degrees of $v_i \in \Gamma(R)$ so that $d_i \geq d_2 \geq \cdots \geq d_n$. The first Zagreb index is the sum of the squares of degrees of the vertices and the second Zagreb index is the sum of the products of the degrees of the pairs of adjacent vertices. We denote the first and second Zagreb indices of $\Gamma(R)$ by $Z_1(\Gamma(R))$ and $Z_2(\Gamma(R))$ respectively. Therefore,

$$Z_1(\Gamma(R)) = \sum_{i=1}^{n} (deg(v_i))^2,$$

$$Z_2(\Gamma(R)) = \sum_{i,j=1}^{n} (deg(v_i))(deg(v_j))$$

where v_i, v_j are adjacent. The Zagreb indices were introduced in [25] and given an elaboration in [26]. Fundamental properties of the indices were summarized in [43]. We obtain the bounds on $Z_1(\Gamma(R))$ and $Z_2(\Gamma(R))$ of classes of the finite rings discussed in Chapter 4 in terms of number of vertices, number of edges, minimum degree and maximum degree. In the sequel we shall require the following results:

Lemma 5.4.1. *[18] Let G be a graph of order n with m edges and d_i be the degree of vertex i then*

$$\sum_{i=1}^{n} d_i^2 \leq m(\frac{2m}{n-1} + n - 1).$$

Lemma 5.4.2. *[6] For positive real numbers $a_1, a_2, \cdots, a_r$, $A^{\frac{1}{2}} \geq B^{\frac{1}{r-1}}$ where*

$$A = \frac{2}{r(r-1)}(a_1a_2 + a_1a_3 + \cdots + a_1a_n + a_2a_3 + \cdots + a_{r-1}a_r)$$

$$B = \frac{1}{r}(a_1a_2 \cdots a_{r-1} + a_1a_2 \cdots a_{r-2}a_r \cdots + a_2a_3 \cdots a_{r-1}a_r).$$

Lemma 5.4.3. *[37] Let $(a) = (a_1, a_2, \cdots, a_r), (b) = (b_1, b_2, \cdots, b_r)$ be two real r-tuples. Then*

$$\sum_{i=1}^{r} a_i^2 \sum_{j=1}^{r} b_j^2 - (\sum_{i=1}^{r} a_i b_j)^2 = \sum_{1 \le i < j \le r} (a_i b_j - a_j b_i)^2.$$

5.4.2 Bounds on First Zagreb Index, $Z_1(\Gamma(R))$ of the Classes of 4-Radical Zero Completely Primary Finite Rings

We make use of Lemma 5.4.1, 5.4.2 and 5.4.3 to generally describe some of the results on the bounds on the first Zagreb index of $\Gamma(R)$ of the classes of 4-radical zero rings.

Proposition 5.4.1. *Let R be classes of rings described in Sections 4.2, 4.3, 4.4 and 4.5 and $\Gamma(R)$ be the zero divisor graph with m edges such that $\mid V(\Gamma(R)) \mid = p^{(h+(k-1))r} - 1$. If $\Delta(\Gamma(R))$ and $\delta(\Gamma(R))$ are the maximum and minimum degrees of $\Gamma(R)$ respectively, then for any $r, k \in \mathbb{Z}^+$, p prime, and h is the dimension of $R'-$module U,*

(i) $Z_1(\Gamma(R)) \ge \frac{((\Delta(\Gamma(R)))^2+(2m-\Delta(\Gamma(R))))^2}{(p^{(h+(k-1)r)}-2)} + \frac{2(p^{(h+(k-1))r}-3)}{(p^{(h+(k-1)r)}-2)^2} . (\Delta_2(\Gamma(R)) - \delta(\Gamma(R)))^2$, *where* $\Delta_2(\Gamma(R))$ *is the second maximum degree of* $\Gamma(R)$.

(ii) $Z_1(\Gamma(R)) \le 4m^2 + 2((\Delta(\Gamma(R)))^2 - 4m((\Delta(\Gamma(R))) - ((p^{(h+(k-1)r)} - 2)((p^{(h+(k-1)r)} - 3))[\frac{T(\Gamma(R))}{(p^{(h+(k-1)r)}-2)\Delta(\Gamma(R))}(I(\Gamma(R))) - \frac{1}{\Delta(\Gamma(R))}]^{\frac{2}{p^{(h+(k-1)r)}-3}}$.

Proof. (i) From Lemma 5.4.3, set $r = p^{(h+(k-1)r} - 2, a_i = d_{i+1}, b_i = 1, i = 1, 2, \cdots r$ which results to

$(p^{(h+(k-1))r} - 2)(\sum_{i=2}^{p^{(h+(k-1))r}-1} d_i^2) - (\sum_{j=2}^{p^{(h+(k-1))r}-1} d_j^2)^2 = \sum_{i,j=2}^{p^{(h+(k-1))r}-1)} (d_i - d_j)^2$. This results to

$(p^{(h+(k-1))r})(Z_1(\Gamma(R)) - (\Delta(\Gamma(R)))^2 = (2m - (\Delta(\Gamma(R)))^2 + \sum_{i,j=2}^{p^{(h+(k-1))r}-1} (d_i - d_j)$.

Now,

$\sum_{i,j=2}^{p^{(h+(k-1))r}-1} \mid d_i - d_j \mid = (p^{(h+(k-1))r} - 3)d_2 - \sum_{i=3}^{p^{(h+(k-1))r}-1} d_i +$
$\sum_{i,j=3}^{p^{(h+(k-1))r}-2} \mid d_i - d_j \mid + \sum_{i=3}^{p^{(h+(k-1))r}-4} d_i - (p^{(h+(k-1))r} - 4)d_{p^{(h+(k-1))r}-1} =$
$(p^{(h+(k-1))r} - 3)(d_2 - d_{p^{(h+(k-1))r}-1}) + \sum_{i,j=3}^{p^{(h+(k-1))r}-2} \mid d_i - d_j \mid \ge (p^{(h+(k-1))r} - 3)(\Delta_2(\Gamma(R) -$

$(\delta(\Gamma(R)))(***)$. which results to

$$\frac{\sum_{i,j=2}^{p^{(h+(k-1))r}-1}(d_i-d_j)^2}{\frac{(p^{(h+(k-1))r}-2)(p^{(h+(k-1))r}-3)}{2}} \geq \frac{\sum_{i,j=2}^{p^{(h+(k-1))r}-1} \mid d_i - d_j \mid}{\frac{(p^{(h+(k-1))r}-2)(p^{(h+(k-1))r}-3)}{2}}.$$

From this we obtain

$$\sum_{i,j=2}^{p^{(h+(k-1))r}-1}(d_i-d_j)^2 \geq \frac{2}{(p^{(h+(k-1))r}-2)(p^{(h+(k-1))r}-3)}(\sum_{i,j=2}^{p^{(h+(k-1))r}-1} \mid d_i - d_j \mid^2)$$

which together with $(***)$ gives
$\sum_{i,j=2}^{p^{(h+(k-1))r}-1}(d_i-d_j)^2 \geq \frac{2(p^{(h+(k-1))r}-3)}{p^{(h+(k-1))r}-2}(\Delta_2(\Gamma(R))-\delta(\Gamma(R)))^2$ which is simplified to give the result as desired.

(ii) We let the simple topological index of $\Gamma(R)$ be $T(\Gamma(R)) = \prod_{i=1}^{p^{(h+(k-1))r}-1} d_i$ and the inverse degree of $\Gamma(R)$ be $I(G) = \sum_{i=1}^{p^{(h+(k-1))r}-1} \frac{1}{d_i}$.

Set $r = p^{(h+(k-1))r} - 2, a_i = d_{i+1}, i = 1, 2 \cdots, r$. Making use of Lemma 5.4.3, we have

$$\sum_{i,j=2}^{p^{(h+(k-1))r}-1} d_i d_j \geq$$

$$\frac{(p^{(h+(k-1))r}-2)(p^{(h+(k-1))r}-3)}{2}[\frac{1}{p^{(h+(k-1))r}-2}\prod_{j=2}^{p^{(h+(k-1))r}-2} d_j \sum_{i=2}^{p^{(h+(k-1))r}-1}\frac{1}{d_i}]^{\frac{2}{(p^{(h+(k-1))r}-3)}} =$$

$$\frac{(p^{(h+(k-1))r}-2)(p^{(h+(k-1))r}-3)}{2}[\frac{1}{(p^{(h+(k-1))r}-2)(\Delta(\Gamma(R))}\prod_{j=2}^{p^{(h+(k-1))r}-2} d_j(\sum_j \frac{1}{d_j}-\frac{1}{\Delta(\Gamma(R))})]^{\frac{2}{(p^{(h+(k-1))r}-3)}}.$$

$$= \frac{(p^{(h+(k-1))r}-2)(p^{(h+(k-1))r}-3)}{2}[\frac{T(\Gamma(R))}{(p^{(h+(k-1))r}-2)(\Delta(\Gamma(R)))}(I(\Gamma(R))-\frac{1}{\Delta(\Gamma(R))}]^{\frac{2}{(p^{(h+(k-1))r}-3)}}.$$

From this we have that

$$\sum_{i,j=2}^{p^{(h+(k-1))r}-1}(d_i-d_j)^2 =$$

$$p^{(h+(k-1))r}-3\sum_{i=2}^{p^{(h+(k-1))r}-1} d_i^2 - 2\sum_{i,j=2}^{p^{(h+(k-1))r}-1} d_i d_j \leq$$

$$(p^{(h+(k-1))r}-3)(Z_1(\Gamma(R))-\Delta(\Gamma(R))(p^{(h+(k-1))r}-2)(p^{(h+(k-1))r}-3)\times$$

$$[\frac{T(\Gamma(R))}{(p^{(h+(k-1)r)}-2)\Delta(\Gamma(R))}(I(\Gamma(R)))-\frac{1}{\Delta(\Gamma(R))}]^{\frac{2}{p^{(h+(k-1)r)}-3}}.$$

Which describes the upper bound on $Z_1(\Gamma(R)$. □

Proposition 5.4.2. *Let R be classes of rings described in Sections 4.2, 4.3, 4.4 and 4.5 and $\Gamma(R)$ be the zero divisor graph with m edges such that $\mid \Gamma(R) \mid = p^{(h+(k-1))r} - 1$. If $\Delta(\Gamma(R))$ is the maximum degree of each $v_i \in \Gamma(R)$ then,*

$$Z_1(\Gamma(R)) \leq (p^{(h+(k-1))r})m - \Delta(\Gamma(R))((p^{(h+(k-1))r} - 1) - \Delta(\Gamma(R))) + \frac{2(m - \Delta(\Gamma(R)))}{p^{(h+(k-1))r} - 3}$$

Proof. Let v_1 be a vertex of maximum degree in $\Gamma(R)$ and $H = \{v_{i1}, v_{i2,\cdots,v_{i\Delta(\Gamma(R))}}\} \subseteq V(\Gamma(R))$ be the vertex set consisting of vertices adjacent to v_1. Let $\Gamma_{v_1}(R) = \Gamma(R) - v_1$ be the induced subgraph of $\Gamma(R)$ obtained by removing a vertex of minimum degree d'_i. Then we have,

$$d'_i = \begin{cases} d_{i-1}, & v_i \in H; \\ d_i, & v_i \in V(\Gamma(R)) \setminus H \end{cases}$$

Since $\Gamma_{v_1}(R)$ has $p^{(h+(k-1))r} - 2$ vertices and $m - \Delta(\Gamma(R))$ the maximum degree of a vertex in the subgraph, then by Lemma 5.4.1 we obtain

$$\sum_{v_i \in V(\Gamma(R))} d_i^2 = 2(m - \Delta(\Gamma(R))),$$

Therefore

$$\sum_{v_i \in V(\Gamma(R))} d_i^2 = \sum_{v_i \in V(\Gamma(R))} 2(m - \Delta(\Gamma(R))) \leq (m - \Delta(\Gamma(R)))[\frac{2((m - \Delta(\Gamma(R))))}{p^{(h+(k-1))r} - 3} + (p^{(h+(k-1))r} - 3)].$$

Now,

$$Z_1(\Gamma(R))) = (\Delta(\Gamma(R)))^2 + \sum_{v_i \in H} d_i^2 + \sum_{v_i \in V(\Gamma(R)))\setminus H, i \neq 1} d_i^2 =$$

$$(\Delta(\Gamma(R)))^2 + \sum_{v_i \in H} (d'_i + 1)^2 + \sum_{v_i \in V(\Gamma_{v_1}(R)))\setminus H} d'^2_i =$$

$$(\Delta(\Gamma(R)))^2 + \Delta(\Gamma(R)) + 2\sum_{v_i \in H} d'_i + \sum_{v_i \in V(\Gamma_{v_1}(R)))} d'^2_i, \mid H \mid = \Delta(\Gamma(R))$$

$$\leq (\Delta(\Gamma(R)))^2 + \Delta(\Gamma(R)) + 2\sum_{v_i \in V(\Gamma_{v_1}(R)))} d'_i + \sum_{v_i \in V(\Gamma_{v_1}(R)))} d'^2_i$$

$$\leq (\Delta(\Gamma(R)))^2 + \Delta(\Gamma(R)) + 4(m - \Delta(\Gamma(R))) + (m - \Delta(\Gamma(R)))[\frac{2((m - \Delta(\Gamma(R))))}{p^{(h+(k-1))r} + 2}].$$

□

5.4.3 Bounds on the Second Zagreb Index, $Z_2(\Gamma(R))$ of the Classes of 4-Radical Zero Completely Primary Finite Rings

Using the previous results obtained in Propositions 5.4.1 and 5.4.2, we generally present some results on the upper and lower bounds on the second Zagreb index $Z_2(\Gamma(R))$ from the maximum and minimum degrees of $\Gamma(R)$ with m-edges as follows.

Proposition 5.4.3. *Let R be classes of rings described in Sections 4.2, 4.3, 4.4 and 4.5 and $\Gamma(R)$ is the zero divisor graph with m edges such that $\mid V(\Gamma(R)) \mid = p^{(h+(k-1))r}-1$ and that $\Delta(\Gamma(R)))$ and $\delta(\Gamma(R))$ are the maximum and minimum degrees of $\Gamma(R)$ respectively. Then for any prime integer p, positive integers r, k, and h the dimension $R'-$module U, we have*

(i) $Z_2(\Gamma(R)) \geq 2m^2 - m(p^{(h+(k-1))r} - 2)\Delta(\Gamma(R)) + \frac{1}{2}(\Delta(\Gamma(R)) - 2)[(\Delta(\Gamma(R)))^2 + \frac{(2m-\Delta(\Gamma(R)))^2}{p^{(h+(k-1))r}-2} + \frac{2(p^{(h+(k-1))r}-3)}{(p^{(h+(k-1))r}-2)^2}(\Delta(\Gamma(R)) - \delta(\Gamma(R)))^2]$.

(ii) $Z_2(\Gamma(R)) \geq 2m^2 - m(p^{(h+(k-1))r} - 2)\delta(\Gamma(R)) + \frac{1}{2}(\delta(\Gamma(R)) - 1)[m(p^{(h+(h-1))r}) - \Delta(\Gamma(R))(p^{(h+(h-1))r} - \Delta(\Gamma(R))) + \frac{2(m-\Delta(\Gamma(R)))^2}{p^{(h+(k-1))r}-3}]$.

Proof. (i) Given that μ is the average distance of the vertices adjacent to $v_i \in V(\Gamma(R))$, we have that

$$Z_2(\Gamma(R))) = \frac{1}{2}\sum_{i=1}^{p^{(h+(k-1))r}-1} d_i^2\mu.$$

We have that

$$\frac{1}{2}\sum_{i=1}^{p^{(h+(k-1))r}-1} d_i[2m - d_i - ((p^{(h+(k-1))r} - 1) - d_i - 1)\Delta(\Gamma(R))] \leq$$

$$Z_2(\Gamma(R)) \leq \frac{1}{2}\sum_{i=1}^{p^{(h+(k-1))r}-1} d_i[2m - d_i - ((p^{(h+(k-1))r} - 1) - d_i - 1)\delta(\Gamma(R))]$$

where $2m^2 - (p^{(h+(k-1))r} - 2)m\Delta(\Gamma(R)) + \frac{1}{2}(\Delta(\Gamma(R)) - 1)Z_1(\Gamma(R))) \leq$

$$Z_2(\Gamma(R))) \leq 2m^2 - (p^{(h+(k-1))r} - 2)m\,\delta(\Gamma(R)) + \frac{1}{2}(\delta(\Gamma(R)) - 1)Z_1(\Gamma(R))).$$

The inequality on the right hand side holds if and only if for every v_i,
$d_i = p^{(h+(k-1))r} - 2$ or $d_j = \delta(\Gamma(R))$ for every v_i if v_i is non adjacent to v_j in $\Gamma(R)$.
Proof for (ii) follows from Proposition 5.4.2. □

CHAPTER SIX

CONCLUSION AND RECOMMENDATIONS FOR FURTHER RESEARCH

In this chapter, we conclude the thesis and highlight gaps which may be considered for future research.

6.1 Conclusion

The main aim of this research was to investigate the structures of matrices and indices of zero divisor graph $\Gamma(R)$ of classes of 3-radical zero and 4-radical zero completely primary finite rings. This has been attained in different chapters due to differences in the ring structures and choices of invariants involved. In Chapter Three, we looked at 3-radical zero completely primary finite ring with the Jacobson radical J satisfying $(J(R))^3 = (0),\ (J(R))^2 \neq (0)$. In this case, we began by looking at the general construction of R with $R'-$modules U and V whose dimensions are s and t respectively such that $t = \frac{s(s+1)}{2}$ for a fixed s. In every characteristic p, p^2 and p^3, we determined the structures of the zero divisors, isolated them and constructed their graphs then obtained the graph geometric properties such as the order, girth, diameter,completeness, minimum and maximum degrees among others. These were illustrated in Propositions 3.3.1, 3.3.2, 3.4.1, 3.4.2, 3.4.6, 3.4.7 and 3.5.1. After characterizing the graphs, adjacency, Laplacian and distance matrices of these graphs were constructed and their algebraic properties such as the trace, order, rank, determinant and the eigenvalues established as the main results of the Chapter demonstrated in Propositions 3.3.3, 3.3.4, 3.3.5, 3.3.6, 3.4.3, 3.4.4, 3.4.5, 3.4.8, 3.4.9 and 3.5.2. It was discovered that for characteristic p, the rank of Laplacian and distance matrices were $p^{(\frac{((s^2+3s)}{2})r} - 2$ and $p^{(\frac{((s^2+3s)}{2}-1)r}$ respectively. For $p \geq 2$, the matrices considered were found to be singular and symmetric. Their spectrum had a 0-eigenvalue and other eigenvalues were of different multiplicities apart from the adjacency matrix whose 0-eigenvalue had a multiplicity of $2p^r-1$ for $p=2$ and $p^{2(\frac{((s^2+3s)}{2}-1)r}-1$ for $p\geq 3$. We also characterised matrices of the rings of characteristic p^2 for cases where $pu_i=0$ and $pu_i\neq 0$ respectively. For both cases, the adjacency and Laplacian matrices were singular and symmetric and for the case where $pu_i=0$, the Laplacian matrix had a rank of $p^{(\frac{((s^2+3s+2)}{2})r}-2$ and its trace was $2p^{(\frac{(2(s^2+3s)}{2})r}-2p^{(\frac{(2(s^2+3s)}{2})r}-2p^{2(\frac{((s^2+3s+2)}{2})r}-p^{(\frac{((s^2+3s)}{2})r}+1$. The non-zero eigenvalues had multiplicities of $p^{(\frac{((s^2+3s)}{2})r}-1$ and $p^{(\frac{((s^2+3s)}{2})r}-2$ for the Laplacian and adjacency matrices respectively and p^r+1 for the distance matrices. For the case where $pu_i\neq 0$, the adjacency and distance matrices were found to have the same multiplicity of $p^{(\frac{((s^2+3s+2)}{2})r}-2$ for the non-zero eigenvalues and distance matrix had $p^{(\frac{((s^2+3s+2)}{2})r}-1$ non-zero eigenvalues. When characteristic of R is p^3 the adjacency and Laplacian matrices were singular and the ranks were established to be $p^{(\frac{((s^2+3s+2)}{2})r}+2$ and $p^{(\frac{((s^2+5s+2)}{2})r}+p^{(\frac{((s^2+3s+2)}{2})r}+1$ respectively. The $0-$eigenvalues for Laplacian matrix and non-zero eigenvalues for the adjacency matrix were established to have the same multiplicity of $p^{(\frac{((s^2+3s+2)}{2})r}-2$. In Chapter Four, this thesis considered matrices of the zero divisor graphs of 4-radical zero completely primary finite rings with Jacobson radical J such that $(J(R))^4=(0),\ (J(R))^3\neq(0)$. The structures of zero divisors were determined, the zero divisors isolated, zero divisor graphs constructed and the matrices were formulated from the graphs. Both algebraic and spectral properties of the matrices were analysed and the results presented as demonstrated in Propositions 4.2.1, 4.2.2, 4.2.3, 4.3.1, 4.3.2, 4.3.3, 4.4.1, 4.4.2, 4.5.1 and 4.5.2. It was noticed that the distance matrices were of full rank.

In Chapter Five, we looked at the graph indices and some general graph properties exhibited by the matrices discussed in Chapter Three and Chapter Four. We began by studying the general properties regarding the product and the sum of eigenvalues in Proposition 5.1.1. Further, a relationship between the graph nullities with respect to the adjacency matrix and the multiplicities of 0-eigenvalues was analysed in Propositions 5.1.2, 5.1.3 and 5.1.4. The general algebraic properties of the matrices from an induced subgraph resulting from removal of a vertex of maximum degree was also discussed and the results presented in Propositions 5.1.5 and 5.1.6. In Propositions 5.1.7, 5.1.8 and Lemma 5.1.1, we established some outcomes from raising the matrices to some finite power. The cases considered were for the adjacency matrices for both $\Gamma(R)$ and its induced subgraph. It was found that if q is any prime integer such that $dim(\Gamma(R)) = q$, then $([A]_{p^{(h+(k-1))r}})^q + ([A]_{p^{(h+(k-1))r}})^{q-1}$ has non-zero entries. The results on the indices of $\Gamma(R)$ were also established beginning from the binding numbers in Propositions 5.2.1, 5.2.2, 5.2.3 and 5.2.4. Further, the Wiener index, average distance and average disorder numbers were also generalized in Propositions 5.3.1, 5.3.2, 5.3.3, 5.3.4, 5.3.5 and 5.3.6. Finally, we generally considered the bounds on the first and second Zagreb indices of $\Gamma(R)$ in relation with $\Delta(\Gamma(R))$ and $\delta(\Gamma(R))$ of the rings in Chapters Three and Four. The results were summarized in Propositions 5.4.1, 5.4.2 and 5.4.3.

6.2 Recommendations

Having studied the matrices and indices of the zero divisor graphs of classes of 3-radical zero and 4-radical zero completely primary finite rings, some gaps still exist to be considered for further research. We therefore recommend a further research on the following:

(i) Matrices and indices of completely primary finite rings satisfying $(J(R))^5 = (0),\ (J(R))^4 \neq (0)$.

(ii) Matrices and indices from Mulay's zero divisor graph $\Gamma_E(R)$.

(iii) Matrices and indices of zero divisor graphs determined by annihilator ideals.

(iv) A study on other graph invariants and indices on the same classes of 3-radical zero and 4-radical zero completely primary finite rings.

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