A CHARACTERIZATION OF FIRST COUNTABLE HAUSDORFF SPACES USING BASES, MORPHISMS AND RANKS

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TITLE PAGE

DECLARATION

This research thesis is my original work prepared with no other than the indicated sources and support and has not been presented elsewhere for a degree or any other award.

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We the undersigned certify that we have read and hereby recommend for acceptance of Masinde Muliro University of Science and Technology a research thesis entitled, **"A Characterization of First Countable Hausdorff Spaces Using Bases Morphisms and Ranks"**

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DEDICATION

To my loving and supportive parents Mr. James Olusala Masunde and Mrs Rael Violet Akhaule, my siblings Abbysai, Phoebe, Jacktone and Esther I dedicate this work for their support and encouragement throughout the course. Without their support it may not have been easier to accomplish this.

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ABSTRACT

The study of separation of axioms is very important in determining classes of Hausdorff spaces. Much of the recent works on the classification of Hausdorff spaces have considered a characterization paradigm of separability, compactness, orderability among others. The characterization of Hausdorff spaces using bases is also a fundamental concept in topology. Indeed the idea of minimal generative bases and their constructions using embedments of global bases is new but useful in presenting intrinsic properties of subspaces of Hausdorff spaces. The main objective of the study was to characterize Hausdorff topological spaces using bases, morphisms and ranks. Specifically, the research aims to construct and analyze quotient spaces of first countable Hausdorff topological spaces, identify minimal generative bases and sub-bases within these spaces, and determine the Cantor-Bendixson rank of Hausdorff spaces, which provides insight into their ordinal structures. This study applies Tietze's Extension Theorem and relevant lemmas for characterizing completely Hausdorff spaces. Through the construction of quotient images and application of separation axioms, it is shown that minimal generative bases are critical in presenting intrinsic properties of Hausdorff spaces and enabling a structured approach to continuity and homeomorphism. The characterization of Hausdorff spaces using minimal bases revealed that a topological space X is Hausdorff if and only if there exists a minimal generative base B_M for X such that for any subfamily B'_M of B_M that is also a base for X, every element of B_M can be expressed as a union of elements from B'_M. This result provides an important tool for studying the properties of Hausdorff spaces and their relationship to other topological spaces. Furthermore, the Cantor-Bendixson rank reveals that under the continuum hypothesis, the rank of the real line $\mathbb{R}$ is the first uncountable ordinal. This characterization has significant implications for topology, providing a concise framework for analyzing bases, morphisms, and ranks within Hausdorff spaces. Such a structured approach simplifies the study of Hausdorff spaces, enhancing our understanding of their complex relationships with other topological spaces. Future research could expand this analysis to examine further applications in functional analysis and differential topology, potentially extending the use of minimal bases and ranks in understanding continuity, compactness, and convergence in broader mathematical contexts.

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NOTATIONS AND ABBREVIATIONS

(X, τ)	:	Topological space
CB_R or C	:	Cantor-Bendixson Rank
ϑ	:	Linearly ordered topological space
ξ	:	Generalized space
F	:	Complete set
π	:	Finite subsets
$\wp$	:	Domain of a complete space
$\propto$	:	Range of a complete space
$\mathbf{B}$	:	Basis of a topology
$\mathbb{N}$	:	The set of natural numbers
Σ	:	Unit sphere
B_M	:	Minimal base
S_M	:	Minimal sub-base
N	:	Neighbourhood
ω_1	:	First uncountable ordinal
N_{S_M}	:	Minimal set containing x
LOTS	:	Linearly Ordered Topological Space
GO	:	Generalized Ordered Space
CB	:	Cantor-Bendixson
X_{Gn}	:	Space of orders
$ht(x)$	:	Height of a topological space

CHAPTER ONE

INTRODUCTION

In this chapter, we give a brief mathematical background, definitions and the concepts that have been useful in our research.

1.1 Background of the Study

Hausdorff spaces are named after a German Mathematician Felix Hausdorff who was one of the founders of Topology. His original definition of a Hausdorff space included Hausdorff condition as an axiom. In [19], Hausdorff introduced the concept of separation axioms which induces the Hausdorff space. The notion of separation as proposed in [19] played an important role in the birth of General Topology as a branch of mathematical analysis. Porter and Thomas [35], investigated H-closed and minimal Hausdorff spaces by considering a collection of properties of a compact Hausdorff space X embedded in a Hausdorff space Y and showed that the image of X is a closed subspace of Y. This was a continuation of Alexandroff and Urysohns' work [1] where the property of compact Hausdorff space was first investigated. In [26], Katetov studied the minimal Hausdorff spaces and discovered that the space was H-closed and semi-regular. In order to study some topological properties like compactness, completeness, limits of sequences, nets and well-behaved functions among others, there is need to invoke the Hausdorff condition which states that for any two disjoint points in (X, τ), there exists neighbourhoods of each point which are disjoint from each other, that is, (X, τ) is said to be a Hausdorff space if for every $x, y \in (X, \tau), x \neq y$, $\exists$ open neighbourhoods G, H such that $x \in G$ and $y \in H$ and $G \cap H = \Phi$ [22]. Therefore, the Hausdorff property has made it easier to study other topological properties like compactness, connectedness, completeness, separability and countability.

On the other hand, Hausdorff [19] introduced and enriched classes of topological spaces called the first countable topological spaces that are characterized using coutability axioms. Indeed, a space X is said to be first-countable if each point $x \in X$ has a countable neighborhood basis, a sequence $N_1, N_2 \cdots$ of neighborhoods of x such that for any neighbouhood N of x there exists an integer i with N_i contained in N. Since every neighborhood of any point contains an open neighborhood of that point, the neighborhood basis can be chosen without loss of generality to consist of open neighborhoods. Assuming that any continuous image of Hausdorff space X is a generalized ordered space, then X is homeomorphic to a countable successor ordinal with order topology [8]. Therefore, there is an intrinsic relationship in the characteristics of separation, countability and ordinal ranks of bases.

The scattered spaces form good ambient spaces for the study of ordinal ranks, their properties and their relationship with first countable Hausdorff spaces. A topological space X is scattered if and only if $x^{(\alpha)}$ is empty for some ordinal α [39]. A space X is scattered if every non-empty subset S of X has an isolated point in S. Fleissner,Vladimir and Yengulap in [16] proved that the complement of every countable subset of a compact space is subcompact for the case of linearly ordered compact space and also for the w-monotholic compact spaces. Every scattered space has a dense set of isolated point and so it is pseudo-complete. Although an example of a second countable space shows that having a dense set of isolated points need not to imply sub-compactness. Any second countable scattered space is Čech-complete space [28].

The notions of limit point and derivated space were both introduced by George Cantor [9]. Cantor demonstrated that representing a function as a trigonometric series is unique for a set, except for a finite Cantor ranked set that needs to be excluded. From the study, it is interesting to note that the class of topological spaces can be naturally transformed into a semi-ring, where Cantor's derivative becomes a derivation.

For ordinal numbers; α, the α^{th} Cantor-Bendixson derivative of a topological space X is defined by transitive induction as follows:

1. $X^0 = X$
2. $X^{\alpha+1}(X^{(\alpha)})'$
3. $X^Y = \bigcap_{\alpha<Y} X^{(\alpha)}$

A topological space is said to be a Linearly ordered topological space if there exists a linear ordering $\leqslant x$ on the set X such that a basis of topology λ^x on x consists of open convex subsets. Robert and Arkady [7] proved that every Hausdorff quotient image of a first countable Hausdorff topological space X is a linearly ordered topological space if and only if X is a metrizable space which is the union of a discrete subspace and a compact countable subspaces. They also characterized σ-compact spaces, locally compact space and separable spaces every quotient of which is a LOTS.

1.2 Basic Concepts

In this section, we present the basic definitions of some terms used in this study. Most of the definitions ar standard and can be found in [10].

Definition 1.2.1. *A topological space is said to be* ***scattered*** *if every non-empty subspace has an isolated point.*

Definition 1.2.2. *A point $x \in X$ is said to be an* ***isolated point*** *of a subset $S \subset X$ if x is an element of S and there exists a neighborhood of X which does not contain any other point of S. An element S is an isolated point of S if and only if its not a limit point of S.*

Definition 1.2.3. *A set A is said to be* ***transitive*** *if every element of A is itself an element of A*

Definition 1.2.4. *A natural number that is considered as an indication of the position in a sequence of objects is referred to as an* ***ordinal number****. Let α be a set; α is an ordinal number if and only if it satisfies the following conditions:*

(i) α *is a transitive set*

(ii) $E \upharpoonright \alpha$ *strictly well-orders* α

The class of all ordinals is denoted as O_n

Definition 1.2.5. *A Topological space (X, τ) is said to be* ***Hausdorff space*** *if for any two distinct points say x, y and $x \neq y$, $\exists$ two disjoint open sets G, H such that $x \in G$ and $y \in H$ and $G \cap H = \emptyset$*

Definition 1.2.6. *A set $S \subset R$ is called* ***compact*** *if every sequence that converges to a point S has a sub sequence that converges to a point in S*

Definition 1.2.7. *A topological space is* ***locally compact*** *if every point has a compact neighborhood base consisting of a compact subspace. A locally compact Hausdorff space is a compactum i.e ,$\forall x \in X$. an open neighborhood $\cup_{x \in X} \supseteq X$ contains a compact neighborhood $kx \subset X$*

Definition 1.2.8. *A space is* ***hereditarily disconnected*** *if no subspace is connected, that is, if the components of all points are singletons.*

Definition 1.2.9. *A space is a* ***K-space*** *if a subset F is closed whenever $F \cap K$ is closed in K for every compact set K*

Definition 1.2.10. *A space is* ***sequential*** *if a subset F is closed whenever no sequence in F converges to a point not in F.*

Definition 1.2.11. *A space is* ***Fréchet*** *if for each subset A and point $x \in \overline{A}$, there is a sequence in A converging to x.*

Definition 1.2.12. *A* ***zero-dimensional*** *topological space is a topological space that has dimension zero with respect to one of several inequivalent notions of assigning a dimension to a given topological space.*

Definition 1.2.13. *A point x in a topological space (X, τ) is said to be an* ***accumulation point*** *of a set S if every neighborhood of x contains at least one point of S other than x itself. In other words, x can be approximated by points of S.*

Definition 1.2.14. *A* ***totally disconnected*** *space is a topological space that has only singletons as connected subsets.*

Definition 1.2.15. *A topological space (X, τ) is said to be a* ***First countable space*** *if every point of X has a countable base of open neighbourhood.*

Definition 1.2.16. *A quotient map is a surjective continuous map between topological spaces that respects equivalence relation defined on one of the spaces. A topological space Y is a quotient image of X if there exists a surjection $f : X \to Y$ such that $V \subset C$ is open if and only it's pre-image $f^{-1}(V)$ is open in X.*

Definition 1.2.17. *(X, τ) is said to be normal if for every pair of the disjoint closed sets E and F there exists open sets G_1 and G_2 such that $E \subset G_1$ and $F \subset G_2$ and $G_1 \cap G_2 = \emptyset$.*

Definition 1.2.18. *Let X be a topological space, the* ***Cantor-Bendixson process*** *on X where X' is the set of the isolated point on X such that:*

1. $X^{(0)} = X$ *and* $X^{(1)} = X \setminus X'$

2. *A successor ordinal* $X^{\alpha+1} = X^{\alpha}$ *where* α *is an ordinal*

3. *If* λ *is a limit ordinal,* $X^{\lambda} = \bigcap_{\alpha<\lambda} X^{\alpha}$ *that is* $X^{\lambda} = Colim_{\alpha<\lambda}$

Definition 1.2.19. *Let* X *be a Hausdorff topological space. A* ***Cantor-Bendixson rank*** *of* X *denoted by* $C_R(X)$ *is the minimal or least ordinal* α *such that* X^{α} *is empty that is,* $X^{\alpha} = X^{\lambda}\forall\alpha < \lambda$

Definition 1.2.20. *A compact Hausdorff space* X *of a Cantor-Bendixson rank* α *is said to be* ***Scattered*** *if the space* X^{α} *is equal to an empty set.*

Definition 1.2.21. *A topological space* X *is said to be a* ***perfect hull*** *if it has no isolated point*

Definition 1.2.22. *Let* X *be a topological space and* $x \in X_S$*.* ***Height*** *of* x *denoted by* $ht(x)$ *is defined as the ordinal* α *such that* $x \in X_{\alpha}$ *but* $x \notin X^{\alpha+1}$

Definition 1.2.23. *A* ***first uncountable ordinal*** *denoted as* ω_1 *represents the smallest ordinal that is uncountably infinite.*

That is the smallest ordinal that cannot be put into a 1-1 correspondence with any countable set of ordinals for example the natural numbers.

1.3 Statement of the Problem

The problem of classification of Hausdorff spaces has been studied to an extend with so many results and applications in functional analysis, algebraic topology, general topology and differential algebra. For instance, the characterization of Hausdorff spaces using morphisms [29] is used to prove that the category of compact Hausdorff spaces is equivalent to the category of commutative C^* algebras. Also, the idea of countability axiom is used to characterize topological spaces into its basis. The basis of a first countable space is required to be countable which means that it has a open neighborhood. This is useful when it comes to limits, convergence, and sequences which are useful in analysis and topology. A lot of current studies involving characterization of Hausdorff spaces have focused mostly on open covers and orderability leaving the characterization of Hausdorff spaces using the ranks and bases fairly untouched. In particular, the characterization of Hausdorff spaces using ranks and bases is still an area of interest despite the attempts giving general notions on bases and their constructions over general topological spaces. This study has therefore classified and characterized Hausdorff spaces using the ranks and bases.

1.4 Objectives of Study

1.4.1 Main Objective

The main objective of this study was to characterize Hausdorff topological spaces using bases, morphisms and ranks.

1.4.2 Specific Objectives

The specific objectives of this research were:

(i) To construct and characterize the quotient spaces of first countable Hausdorff topological space over some topological spaces .

(ii) To determine classes of minimal generative bases and sub bases of the spaces constructed.

(iii) To determine the relation between a Hausdorff space and the Cantor-Bendixson rank.

1.5 Research Methods

The following methods have been used in the study:

i. **Tietze's Extension Theorem** [12]

Let X be a topological space. Then the following are equivalent:

(a) X is completely Hausdorff.

(b) X is Hausdorff and compactly embedded.

ii.

Lemma 1.5.1. *[12]*

Let X be a topological space. Then the following are equivalent:

(a) X is completely Hausdorff.

(b) For each $a, b \in X$ with $a \neq b$ there exist continuous functions $f, g : X \to \mathbb{R}$ such that $a \in Z(f)^\circ, b \in Z(g)^\circ$ and $Z(f) \cap Z(g) = \emptyset$.

iii. (**Mazurkiewicz-Sierpinski** [33]).

Theorem 1.5.1. *Every countable compact Hausdorff space is homeomorphic to some well-ordered set with the order topology.*

1.6 Significance of the Study

Characterizing minimal generative bases in first countable Hausdorff spaces is an important concept in topology and can simplify the study of these spaces by providing a concise description of the topology. Both quotient maps and images are essential concepts in mathematics and can be applied in various areas, including topology, algebra, and analysis, to create new spaces and study their properties. The Cantor-Bendixson rank is a useful tool for characterizing and understanding the structure of subsets in topological spaces, and it is particularly relevant in descriptive set theory, where it helps classify sets according to their complexity. It provides a way to measure the "size" or "rank" of closed sets within a space, which can be a useful concept in various mathematical contexts.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

In this chapter, we give an exhaustive review of some of the related studies on Hausdorff spaces, Hausdorff quotient image, scattered spaces and first countable topological spaces.

2.2 Hausdorff Spaces

The Hausdorff spaces, named after Felix Hausdorff [19], has been a subject of interest for a quite some time. More research has been done on Hausdorff spaces over the years. Fell [15] carried out a research on Hausdorff topology for the closed subsets of a locally compact non Hausdorff space. Graves [17] looked at the completeness of a Hausdorff space and revealed that the property of completeness carries over to certain spaces of functions whose functional value lies in a complete space (X, τ). He allowed the neighborhood $U\pi\sigma$(F) to consist of all functions of G in j such that $G(qi)$ is in $V\sigma(F(q_i))$ for each q_i in π with a proper definition of the relation $>$ for the complete space. Using the neighborhoods $W\alpha(F)$, it is shown that certain sub-spaces of j are also complete. A set S in the space $\propto$ is totally bounded if for every σ are Hausdorff spaces $\exists$ points $x_1, x_2, ...x_n$ in $\propto$ such that S is contained in $V\alpha x_1 + V\alpha x_2 + ... + V\alpha x_n$. The space jB consists of all totally bounded functions and F is complete. If (X, τ) is a topological space, then the space of all continuous functions f is complete and if (X, τ) has associated itself with a system of neighborhoods, then the space (X, τ) of all uniformly continuous functions of (X, τ) is complete.

Arens and Kelly[3] carried out a research on Characterizations of the space of continuous functions over a compact Hausdorff space. According to their research, the first characterization proceeds via an investigation of the extreme points of the unit sphere Σ of the adjoint and space B^* of the Banach space B under construction. If B is the class of continuous real-valued functions over a compact Hausdorff space X, then the $\|f\| = sup_{x \in X}|f(x)|$. The Reisz-Markoff Saks representation [36] for linear continuous functionals on C is used to proof the class of extreme point of Σ and can be divided into two disjoint closed sets, each of which is homeomorphic to X using the weak topology in B^*

Bing[6]connected a countable Hausdorff spaces with countably many points. Urysohns gave an example of a connected Hausdorff space with only countably many points. Bing [6] used an example that the points of the space are rational points in the plane on or above the x-axis. That is if (a, b) is such a point and $\epsilon > 0, (a + b) + (r + 0)$ then either $\mid r - (a + 6/3, /2 \mid< \epsilon$ or $\mid r - (ab/3| > 2) \mid< \epsilon$ is a neighborhood. To construct geometrically a neighborhood with center (a, b), Bing [6] considered an equilateral triangle with base on the x-axis and apex at (a, b). If $b = 0$ and regard (a, b)as the triangle then (a, b) plus all rational points on the $x-$axis whose distances from a base vertex of the triangle are less than ϵ and is a neighborhood with (a, b). The space therefore satisfies the Hausdorff axiom and it has a property that for each neighborhood there is a point common to their closures and therefore the space is connected.

Ronse and Mohamed [37] used a new approach to discretisize Hausdorff spaces such that the discretization of a compact Hausdorff distance to K is minimal. A mathematical description of Hausdorff discretizing sets which are related to the discretization by dilation was considered by Heijmans and Toet [20] in the cover of discretization studied by Andres [2]. Andres worked on the Hausdorff distance between a compact set and its maximal Hausdorff discretization. From this study its clear that the latter converges (for the Hausdorff metric) to the compact set when the spacing of the discrete grid tends to zero.

Ercan [12] attempted to give a better understanding of characterization of completely Hausdorff spaces. A topological space is said to completely Hausdorff [42] if for every $x, y \in X$ and $x \neq y$, then there exists a continuous function $f : X \to R$ such that $f(x) \neq f(y)$. [12] showed that a topological space is completely Hausdorff if and only if every compact subspace is C_b . A subspace Y of X is said to be C_b embedded if every bounded continuous function $f : Y \to R$ has a continuous extension $\overline{f} : X \to R$. Since every compact subspace of X is C_b embedded then X is compactly embedded. This therefore shows that the completeness of Hausdorff spaces and compactly embedded spaces brings out the notion of equivalence for Hausdorff spaces.

Theorem 2.2.1. *([12]) Let X be a Topological space and Y a subspace of X. Then the following statements are equivalent:*

1. *Y is C_b embedded*

2. *Every pair of completely separated*

This theorem was important as it helped in the characterization of the completely Hausdorff spaces.

Furthermore, the following lemmas were also useful as they characterized completely Hausdorf spaces.

Lemma 2.2.1. *([12]) Let X be a Topological space, then the following statements are equivalent:*

1. *X is completely Hausdorff*

2. *For each $a, b \in X$ with $a \neq b$, there exists a continuous function $f : X \to R$ such that $a \in Z(f)^0$ and $b \in Z(g)^0$ and $Z(f)^0 \cap Z(g)^0 = \emptyset$*

Lemma 2.2.2. *([12]) let X be a topological space. The following are equivalent;*

1. *X is completely Hausdorff*

2. *X is Hausdorff and any two distinct compact subsets of X are completely separated.*

It's clear that the two lemmas addressed completeness of Hausdorff spaces and also separability of topological spaces. Therefore the main results of [12] which is a version of Tietze's Extension Theorem states as follows:

This theorem proves that for a topological space to be completely Hausdorff then it has to be compactly embedded that is the two have to be equivalent and compactly embedded and hence bringing out the notion of equivalence.

However, this deviates from our study since the result focuses mostly on completely Hausdorff spaces whilst our study is fixed in first countable Hausdorff spaces and their characterization using bases and morphism.

Bellomonte in [5] defined a quasi C^*-algebra as a pair $(\mathfrak{U}, \mathfrak{U}_0)$ consisting of a vector space $\mathfrak{U}$ and a $^*-$algebra $\mathfrak{U}_0$ contained in $\mathfrak{U}$ as a subspace such that;

i. U carries an involution $a \mapsto a^*$ extending the involution of $\mathfrak{U}_0$;

ii. $\mathfrak{U}$ is a bi-module over $\mathfrak{U}_0$ and the module multiplication extend the multiplication of $\mathfrak{U}_0$. In particular the associative law holds.

iii. $(xa)^* = x^*a^*$, for every $a \in \mathfrak{U}$ and $x \in \mathfrak{U}_0$

The identity of $(\mathfrak{U}, \mathfrak{U}_0)$ is a unique element $e \in \mathfrak{U}_0$ such that $ae = a = ea \forall a \in \mathfrak{U}$. Quasi *-algebra that have the M of invariant positive sesquilinear forms have numerous topologies related to M which forms every *-representation. This thus leads to the definition of the class of locally convex quasi *-algebra which consists of the family of their bounded elements with respect to M-family that is dense C*-algebra hence has a connection to Topological spaces. The *- representations bring out more useful information about the topological properties of quasi *-algebra such as the *-strong operator and *-weak operator, which are important when it comes to the continuity and compactness in algebra. It is also important when it comes to investigating the structure and behavior of quasi-*algebra. The *-representation helps explore algebraic structures of quasi-algebra which also the establishment and connection between different areas of mathematics.

2.3 Scattered Spaces

A topological space (X, τ) is called scattered or dispersed if its every non-empty subspace has an isolated point [39]. Also, a space is scattered if and only if it is right separated, so $|X| \leq (X)$ for scattered spaces. Soukup [39] showed that a scattered space is hereditarily disconnected and locally compact scattered spaces are totally disconnected and so they are zero-dimensional. A zero-dimensional scattered space need not be normal.

Kannan and Rajagoplan, [25] proved that a scattered zero-dimensional Hausdorff space can be mapped continuously on a one to one, onto a scattered zero-dimensional Hausdorff space of the same weight as its cardinality. They also proved that a countable regular space has a coarser compact Hausdorff topology if and only if its scattered and that a zero-dimensional Lindelof, scattered first countable Hausdorff space admits a scattered compactification.

A Topological space is said to be non-Archimedean if there exists a basis $\mathbf{B}$ for open sets such that if B_1 and B_2 are members of the basis $\mathbf{B}$ with $B_1 \cap B_2 \neq 0$, then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Non-Archimedean P-spaces are scattered spaces if they have a continuous selector. [4]Artico, Marconi ,Moresco and Pelant gave many results for spaces that have a unique accumulation point, non-Archimedean spaces and ultracompact spaces. The P-space is the union of countably many closed sets that are closed and therefore implying that non-archimedean spaces are P-spaces provided every countable subset is closed. This was a continuation of [11] where the existence of a continuous selector in a non-archimedean complete metric space was provided. For every ultracompact scattered space X, there exists a continuous selector on X. Every non-discrete scattered space contains a clopen subspace of the form $S \cup \{P\}$ and therefore when deciding whether a scattered space X has a continuous selector, the knowledge of when there are selectors on spaces of the form $S \cup \{P\}$ is required. Therefore $S \cup \{P\}$ is a strongly zero-dimensional paracompact space and it is not non-archimedean.

Fleissner [16] proved that every scattered space is hereditarily subcompact and finite union of subcompact spaces is subcompact. William, Vladimir and Yengulap [16] also proved that the complement of every countable subset of a compact space is subcompact for the case of linearly ordered compact space and also for the w-monotholic compact spaces. Every scattered space has a dense set of isolated point and so its pseudo-complete. Although an example of a second countable space shows that having a dense set of isolated points need not to imply sub-compactness. Any second countable scattered space is Čech-complete space [28]. Generally, Fleissner, Vladimir, and Yengulap [16] showed that for any countable space X, the following are equivalent:

i X is hereditarily subcompact

ii X is compact

iii X is scattered

2.4 Hausdorff Quotient Image

Ernest [34]characterized the regular quotient images of separable metric spaces. His results were generalized by Strong [41] and used in the characterization of the T_1

quotient images of separable metric spaces and afterwards used to characterize the Hausdorff pseudo-open images of separable metric spaces [41]. Also, Jayanthan and kannan [24] characterized topological spaces for which every quotient is metrizable. But we learn that every continuous image in Hausdorff space of a compact metric space is metrizable [42]. Jayanthan and Kannan [24] showed that every quotient of a metric space is metrizable if and only if the set of all limit points of X is compact. Fedeli and Le Donna [14] looked on when a connected space has a connected pre-image with some additional properties and characterized continuous and quotient images of a paracompact connected space.

Albert, Alessio and Ludovic [13] under appropriate assumptions on the dimensions of the ambient manifold and the reukarity of the Hamilitonian showed that the Mather quotient is small in term of the Hausdorff dimension. They used the completeness of Euler-Lagrange flow that is automatic for Tonelli lagrangians independent of time that is readily adapted to the case where L depends on time, 1-periodic in time and has a complete Euler-lagrange flow as in the work of Mather ([30], [32]).

2.5 First Countable Spaces

According to Gutierres [18] a topological space is *first countable* if there is a countable neighborhood base (or local base) at each of its points. In the study on first countable spaces, Zhu [23] considered generalizations of first countable spaces known as w_k spaces. These spaces are Frechet spaces, w-spaces only when $k = 1, \omega_1, -\infty$. He showed that the $-w_k$ spaces are the images of metric spaces under certain kind of continuous maps known as w_k maps. Frechet spaces and first countable spaces are respectively the images of metric spaces under pseudo-open almost open maps [11]. Zhu [43] proved that w are the images of metric spaces under w-spaces.

A topological space is said to be a Linearly ordered topological space if there exists a linear ordering $\leqslant x$ on the set X such that a basis of topology λ^x on x consists of open convex subsets. Robert and Arkady [7] proved that every Hausdorff quotient image of a first countable Hausdorff topological space X is a linearly ordered topological space if and only if X is a metrizable space which is the union of a discrete subspace and a compact countable subspaces. They also characterized σ-compact spaces, locally compact space and separable spaces every quotient of which is a LOTS.

From the literature, it is evident that the study of first countable Hausdorff spaces has been active to some extend. However, from the for going studies, it is worth noting that little has been done regarding characterization of first countable Topological spaces using bases, morphisms and ranks. In particular, minimal generative bases and sub bases have attracted not much attention despite the fact that bases is a fundamental tool when it comes to first countable spaces. This research therefore characterizes Hausdorff topological spaces using bases, morphisms and ranks while extending the idea of first countable topological spaces to Hausdorff quotient image.

CHAPTER THREE

ON THE CATEGORIES OF BASES AND MORPHISMS OVER HAUSDORFF SPACES

In this chapter we have categorized bases and morphisms over Hausdorff spaces. We have also constructed quotient maps and quotient images over some topological spaces including the metric space (R, d).

The following definition shall be useful in the sequel:

Definition 3.0.1. *A quotient map is a surjective continuous map between topological spaces that respects equivalence relation defined on one of the spaces. A topological space Y is a quotient image of X if there exists a surjection $f : X \to Y$ such that $V \subset C$ is open if and only it's pre-image $f^{-1}(V)$ is open in X.*

3.0.1 Construction of Quotient Maps/Images

In order to construct a quotient map, we need to define an equivalence relation. Let $(\mathbb{R}, d)$ be the metric topological space with the Euclidean topology d and consider an interval $[0, 1] \subset (\mathbb{R}, d)$. Two points x, y can induce an equivalence relation as $x \sim y$ if $x = y$ or if both are irrational. Now given two spaces, (X, τ) and (Y, ρ), the spaces are considered equivalent if there exist two distinct points $x \in X$ and $y \in Y$ inducing the relation equivalence.

Proposition 3.0.1. *Let $[0, 1] \subset (\mathbb{R}, d)$ be given and let X be the set of all equivalence classes with a topology generated by the set $U = [(1/2) - \epsilon, (1/2) + \epsilon]$ with $\epsilon > 0$. Define a map $f : [0, x] \to X; f(x) = [x]$ in $[0, 1]$ whereas $[x]$ denotes the equivalence class containing x. It is clear that the map is surjective as the equivalence class in X is also in $[0, 1]$. $f(x) \sim f(y)$ if and only if $x \sim y$. Thus f respects* *the equivalence relation.*

Proof. Let $U \subset X$ and $u \in U$ such that $u = [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$. Consider a surjection f with $f : [0,1] \to X$. Since $[0, x] \subset [0,1]$, then, $f[0,1] = [x]$ thus

$$f : [0,1] \to X \bigcap_{u \in U} u.$$

It is also clear that $f(0,1) = (0, u)$ whenever $x \in [0,1] \forall u \in X$. Hence $f(u) = u \in U$ (1)

Pre-multiplying equation (1) by f^{-1} gives: $f^{-1}(f(u)) = f^{-1}(u)$ implying that $x = f^{-1}(u) \in [0,1]$ and therefore $f : [0,1] \to X$. From this it is clear that $[0,1] \diagup ker f \cong ImX$. The images of X via f^{-1} lie in $[0,1]$ and they are a representation of quotient images in the pre-image. This means that ker f $=\{0\}$.

Finally we need to show that f is continuous. Let U be an open subset of X. We need to show that $f^{-1}(u)$ is an open subset of $[0,1]$. Since the topology generated by the set $U = \{[(1/2) - \epsilon, (1/2) + \epsilon]\}$, it is enough to show that $f^{-1}(U)$ is open for such a $\{u\}$

Let $x \in f^{-1}(U)$. If $f(x)$ belongs to U it would imply that $[x]$ intersects U. Thus there exists $\epsilon > 0$ such that $[(1/2) - \epsilon, (1/2) + \epsilon]$ intersects U. Since it is already known that $[x]$ is an equivalence class then there exists a point $y \in [x]$ such that $\mid y - 1/2 \mid < \epsilon$ and also since f maps y to [x] , then we have $f(y) \in U$. Thus for any point $x \in f^{-1}(u)$ there exists an $\epsilon > 0$ such that the interval $(x - \epsilon, x + \epsilon)$ is contained in $f^{-1}(U)$ implying that $f^{-1}(u)$ is open and therefore continuous. □

Example 3.0.1. *Let* $(\mathbb{R}, d)$ *be the metric topological space with the Euclidean topology* d *such that* $d(x, y) < r$ $\forall r \in \mathbb{R}^+$ *if and only if* $x \sim y$. *This partitions* $\mathbb{R}$ *into* *equivalence classes that are balls of radius r centred at each point in $\mathbb{R}$. The quotient space obtained here consists of all these balls as points and we endow it with a metric induced.*

If $r = 1$. Then for any two points x, y in $\mathbb{R}$, we get

$$[0] = \{0\},\ [x] = y : d(x, y) < 1,\ [R \diagdown \{x\}] = y : d(x, y) \geqslant 1$$

as the only equivalence classes.

3.1 Separation Axioms

Definition 3.1.1. *A topological space (X, τ) is called a T_1* ***space*** *if it satisfies the T_1-Axiom.*

Let (X, τ) be a T_1 **space**. Then for every pair x, y of distinct points of X there exist subsets $G,\ H$ of X such that $x \in G$ and $x \notin H,\ y \in H$ and $y \notin G$. But $G \cap H \neq \Phi$.

Proposition 3.1.1. *Let (X, τ) be a T_1* ***space****. Then for every pair x, y of distinct points of X there exist subsets $G,\ H$ of X such that $x \in G$ and $x \notin H,\ y \in H$ and $y \notin G$. But $G \cap H \neq \Phi$.*

Proof. Let (X, τ) be T_1 and $p \in X$. We need to show that $\{p\}$ is closed. Hence, we must show that $\{p\}^c$ is open.
Let $q \in \{p\}^c$. So $q \neq p$. Since (X, τ) is T_1 there exists an open set G_q containing q but not p. That is, $q \in G_q$ but $q \neq p$. Now,

$$\{p\}^c = \cup\{q \in X : q \neq x\}.$$

So,

$$\{q\} \in G_q \subseteq \{p\}^c \; \forall q \neq p.$$

Thus

$$\cup\{q\} \subseteq \cup_q G_q \subseteq \{p\}^c.$$

Therefore $\{p\}^c = \cup_q G_q$ which is an open set for it is a union of open sets. Thus $\{p\}^c$ is open. That is, $\{p\}$ is closed.

Conversely, let $\{p\}$ be closed, for each $p \in X$, the space (X, τ) is T_1. Also, let $p,\ q \in X$ and $p \neq q$. Then, $\{p\}^c$ is open, containing q but not p. On the other hand $\{q\}^c$ is an open set containing q but not p. Thus, (X, τ) is T_1. □

3.2 Continuity and Homeomorphism Between Hausdorff Quotient Images

Definition 3.2.1. *Let (X, τ) and (Y, ψ) be topological spaces and $f : X \to Y$ be a function. Let p be any fixed point in X. We say that f is $\tau - \psi$ continuous or continuous at the point p if each open set $H \in U$ containing $f(p)$ there is an open set containing p such that $f(u) \subseteq H$. that is $U \subseteq f^{-1}(H)$.*

Proposition 3.2.1. *Let (X, τ) and (Y, ψ) be topological spaces and let $f : X \to Y$ be a function. Let B be base for ψ and S be a sub-base for ψ. Then the following statements are equivalent:*

(i) f *is* $(\tau - \psi)$ *continuous.*

(ii) $f^{-1}(B) \in \tau$ *for each* $\beta \in B$

(iii) $f^{-1}(S) \in \tau$ *for each* $s \in S$

Proof. *(i)* $\Rightarrow$ *(ii)*

Assume that (i) holds, then by definition of continuity of f, the basis $B \subset \psi$ since

ψ is open. Thus any $\beta \in B \Rightarrow \beta \in U$. Since f is continuous $f^{-1}(\beta) \in \tau$ which is (ii).

$(i) \Rightarrow (iii)$

Since $S \subset \psi$, each $s \in S$ belongs to ψ thus by (i) $f^{-1}(s) \in \tau$ for each $s \in S$ which is (iii).

Conversely $(iii) \Rightarrow (ii)$

By $(iii) f^{-1}(s) \in \tau$ for each $s \in S$. Let $H \in \psi$. Since S is a sub-base of ψ which can be written as

$$H = \cup_i (S_{i_1} \cap S_{i_2} \cap ..... \cap S_{i_{n_i}} : n_i \in \mathbb{N}) \Rightarrow f^{-1}(H) = f^{-1}(\cup_i (S_{i_n} \cap ...... \cap S_{i_{n_i}})).$$

Each $f^{-1}(S_{i_k} \in \tau)$. Thus, $f^{-1}(H) \in \tau$ which proves (i). $\square$

Proposition 3.2.2. *Let (X, τ) and (Y, ψ) be topological spaces and let $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) f is $\tau - \psi$ continuous

(ii) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X

(iii) $\overline{f^{-1}(B)} \subseteq f^{-1}\overline{(B)}$ for every subset B of Y.

Proof. For every closed subset F of Y , $f^{-1}(F)$ is a closed subset of X.

Now assume that $(i) \Rightarrow (ii)$ and $f(A)$ is a closed subset of Y. So by (i) it follows that $f^{-1}(\overline{f(A)})$ is a closed subset of X containing A

$\therefore f^{-1}(\overline{f(A)}) \supseteq \overline{A}$ from the condition that suppose $M \supseteq A \Rightarrow M \supseteq \overline{A} \Rightarrow \overline{A} \subseteq f^{-1}(\overline{(f(A))}$ where

Thus

$$f\overline{(A)} \subseteq f(f^{-1}(\overline{f(A))} = \overline{f(A)} \text{ which proves (ii).}$$

$(ii) \Rightarrow (iii)$

By (ii), $f(\overline{(A)} \subseteq \overline{f(A)}$. Let $A = f^{-1}(B)$. Substituting in (ii), we get $f(\overline{f^{-1}(B)} \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}(f(f^{-1}(B) \subseteq (B)$. Therefore, $f^{-1}\{f(\overline{f^{-1}(B)}\} \subseteq f^{-1}\overline{(B)}$. Now $f^{-1}(f(\overline{f^{-1}(B)})) \supseteq \overline{f^{-1}(B)}$ then $\overline{f^{-1}(B)} \subseteq \overline{f^{-1}(B)}$ which is (iii).

Finally, $(iii) \Rightarrow (i)$

By (iii) $f^{-1}(B) \subseteq f^{-1}\overline{(B)}$ for every $B \in Y$. To prove (i), let B be any closed subset of Y to show that $f^{-1}(B)$ is closed in (X, τ). Therefore, $\overline{B} = B$ and $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$. But $\overline{f^{-1}(B)} \supseteq f^{-1}\overline{(B)}$ always.

Thus $\overline{f^{-1}(B)} \supseteq f^{-1}(B)$. That is, $f^{-1}(B)$ is a closed subset of (X, τ) which proves (i) □

Proposition 3.2.3. *Let (X, τ) and (Y, ψ) be topological spaces and $f : X \to Y$ be such that:*

(i) f is a bijection.

(ii) f is $(\tau - \psi)$ continuous.

(iii) f is also continuous

Then f is called a homeomorphism from the topological space (X, τ) to (Y, ψ) and f^{-1} is a homeomorphism from the topological space (Y, ψ) to (X, τ) and the two spaces (X, τ) and (Y, ψ) are said to be homeomorphic written $(X, \tau) \sim (Y, \psi)$

3.3 Countability Axioms

Definition 3.3.1. *Let (X, τ) be a topological space and $p \in X$. A family B_p of open sets containing p is called a **local base** at p if for each open set G containing p there is a member $\beta_p \in B_p$ such that $p \in B_p \subseteq G$.*

Now a topological space (X, τ) is said to be first countable if it satisfies the following axiom of countability:

At each point $p \in X$ there is at most countable family B_p of open sets containing p, that is, $\beta \in B \in G$.

In other words, each point $p \in X$ is associated with an **atmost countable local basis.**

Example 3.3.1.

1) *Take any metric space (X, ρ). Let $p \in X$. Take $B_p = \{\mathcal{N}(P : 1/n), n \in \mathbb{N}\}$. $B_p$is countable.*

If G is open and $p \in G$, then there exists a member $\beta \in B_p$ such that $x \in \beta_p \in G$

$\therefore (X, \rho)$ is a first countable space.

2) *Take X and the discrete topology $\tau = P(X)$. Each $\{p\}$ is open $\forall p \in X$. Take $B_p = \{\{p\}\}$. So B_p is at most countable.*

For each open G containing p, we have $p \in \{\{p\}\} \in G$. Therefore, $(X, P(xX))$ is first countable.

Definition 3.3.2. *A topological space $(X, \tau)$is said to be* ***second countable*** *if it has an at most countable base*

Example 3.3.2. $(\mathbb{R}, \tau)$ *where $\tau = d$ the usual topology on $\mathbb{R}$ is second countable.*

Reason: $\{\mathcal{N}(q : 1/n) : q \in \mathbb{Q}\}$ is an open at most countable neighbourhood.

Proposition 3.3.1. *Let X be a connected Topological space. Then X is connected if its Hausdorff but the converse is not true.*

Proof. Assume that X is not a connected Topological space. If $x \in X$, then we have a neighbourhood $N_x(x, r_1)$ and $N_y(y, r_2)$ such that $x \in N_x(x, r_1)$ and $y \in N_y(y, r_2)$. From the definition of connectedness, if we have two open sets U and V then $U \cap V = \Phi$.

This implies that $N_x(x, r_1) \bigcap N_y(y, r_2) = \Phi$ implying that separation holds which is a contradiction. Hence it is a connected space.

Alternatively, suppose $x \in X$ and suppose that X is not connected. Consider $X = x, y, z$ and $\theta = (x, y)(y, z)$ such that θ is the base of the topology τ meaning that $\theta \in \tau$ and hence $(x, y) \cup (y, z) \in \tau$. This implies that x, y, z covers the entire space in τ This a contradiction and hence the space is connected.

Conversely, suppose X is a Hausdorff Topological space. There are two distinct points x and y such that $x, y \in X$; there exists two open sets G and H such that $x \in U$ and $y \in V$ and $U \cap V = \Phi$. This a contradiction for connectedness since a connected space, the space cannot be split into two disjoint. Hence the converse does not hold

□

Definition 3.3.3. *A first countable Hausdorff space X is a topological space whereby each point has a countable neighborhood and the space also obeys the separation axiom. That is every two distinct points have two disjoint open neighborhoods*

Lemma 3.3.1. *Let X be a first countable Hausdorff space. Then X is a T_1 space. In other words every singleton set is closed.*

Proof. Let X be a first countable Hausdorff space and let $x \in X$ where x is an arbitrary point. We want to show that for every two distinct points say $x, y \in X$ there exists open sets U, V; $x \in U$ and $y \in V$ such that $x \notin V$ and $y \notin U$.

Since X is first countable, for each point $x \in X$ there exists a countable neighborhood basis $B_x = \{U_n\}_{n=1}^{\infty}$ at x where B_n is an open set containing the arbitrary point x. Similarly for each point $y \in X$ there exists a countable neighborhood basis $B_y = \{V_n\}_{n=1}^{\infty}$ at y where B_n is an open set containing the arbitrary point y. Since B_n is an open set containing x and y then there is a neighborhood N of x such that $B_n \subset N \ \forall \ n \in N$. Now consider the singleton sets of $\{x\}$ and $\{y\}$. Defining the two open sequences of open sets we will have;

1. For each n, let $U_n = X \setminus \{y\} \cup \{x\}$

2. For each n, let $V_n = X \setminus \{x\} \cup \{y\}$

Its clear from above that each U_n contains x but not y and same thing happens to V_n. We want to show that the two sequences are of open sets.

Since X is a Hausdorff space, then for every two distinct points $x, y \in X$ there exist two open disjoint sets U, V; $x \in U$ and $y \in V$ such that $U \cap V = \Phi$. Let $x, y \in B_n$ and $U \cap V = \Phi$. This implies that $X \setminus \{x\} \cap V = U \cap V = \Phi : \forall x, y \in X$, and $X \setminus \{y\} \cap U = U \cap V = \Phi : \forall x, y \in X$.

Since $X \setminus \{x\}$ is the set of union of points that do not intersect V_y it then follows that $X \setminus \{x\}$ is open. Similarly $X \setminus \{y\}$ is the set of union of points that do not intersect U_x it then follows that $X \setminus \{y\}$ is open.

Also since U contains x but not y and V contains y but not x for all $x, y \in X$, then x is T_1space. Therefore we have shown that for any point x in the first countable Hausdorff space the singleton set $\{x\}$ and $\{y\}$ are closed and hence satisfying the T_1 separation axiom □

Theorem 3.3.1. *Every Hausdorff space is a T_1 space, but the converse is not true.*

Proof. To prove that every Hausdorff space is a T_1 space, we start by recalling the definitions of these topological properties. A topological space X is termed a **Hausdorff space** (or T_2) if, for any two distinct points x and y in X, there exist disjoint neighborhoods U of x and V of y. On the other hand, a space is called a T_1 **space** if for any two distinct points x and y, there exists a neighborhood of x that does not include y (and vice versa).

Now, let us assume X is a Hausdorff space and consider any two distinct points x and y in X. By the definition of a Hausdorff space, we can find neighborhoods U of x and V of y such that $U \cap V = \Phi$. This implies that y cannot be contained in the neighborhood U of x. Consequently, we have established that every neighborhood of x excludes y. Similarly, since the neighborhoods are disjoint, x cannot be included in the neighborhood V of y. Thus, every neighborhood of y excludes x. Therefore, we conclude that X satisfies the T_1 separation condition.

Next, we demonstrate that the converse is not true by providing a counterexample. Consider the **Sierpiński space**, which consists of two points, $\{a, b\}$, with the topology $\{\Phi, \{b\}, \{a, b\}\}$. In this topology, the singleton set $\{b\}$ is a neighborhood of b that does not contain a, thus satisfying the T_1 condition. However, there are no disjoint neighborhoods around the points a and b; any open set that includes a must also include b. This means that the space does not satisfy the Hausdorff condition.

In conclusion, while every Hausdorff space is indeed a T_1 space due to the ability to separate points with disjoint neighborhoods, the converse fails, as illustrated by the Sierpiński space, which is T_1 but not Hausdorff. □

Lemma 3.3.2. *Let (X, τ) be a Topological space and (Y, φ) be a Hausdorff space. If A is a non-empty subset of X; then $f : X \to Y$ is continuous.*

Proof. If $f : X \to Y$ is continuous, we expect that f is an injection and there exists an open subset $A \subset X$ such that $f(A) \subset y$ Also $\{f^{-1}(y) : y \in Y\}$
$subset A$.

Since X is Hausdorff, Let $A = N_x(x, r)$ where $x \in X$. If there exists $G \subseteq X$: $x \in N_x \subseteq G$. Then A is any neighborhood in X mapped into an open neighborhood V in Y. Since f is continuous, $f^{-1}(V)$ must also be open in X. Moreover, because Y is Hausdorff, this mapping respects the separation of points in X, preserving open sets.

Thus, the continuity of f on a Hausdorff space Y ensures that the preimage of each open subset of Y remains open in X, completing the proof that f is continuous as required. □

Lemma 3.3.3. *Let X be a first Hausdorff countable Hausdorff space. Each mini-*

mal base $B \in X$ is closed and can be embedded.

3.4 Base of the Constructed Spaces

Definition 3.4.1. *Let (X, τ) be a topological space. A sub-family B of τ is called a* ***base or basis*** *for the topology τ if each open set G of X (ie $G \in \tau$) can be expressed as a union of members of B. In other words for each $G \in \tau$ and $x \in G$, there exists some $\beta \in B$ such that $x \in B \subset G$.*

From the definition 3.4.1 above, we have two statements which are equivalent:

(i) Let G be a union of members of B : ie $G = \cup B_i : B_i \in B, i \in \mathbb{N}$.

(ii) Let $x \in G$. Then $x \in B_i$ for some i and $B_i \subset G$ that is $x \in B_i \subset G$

Remark 3.4.1. *If B is a base for the topology τ of X, then B is itself a part of τ ie members of the base B are themselves open sets and subsets of (X, τ).*

Proposition 3.4.1. *Not every collection of subsets of X can serve as a base for a topology on X*

Proof. Consider $X = \{a, b, c\}$ and $\theta = \{\{a, b\}, \{b, c\}\}$. Claim that θ serves as a base for a topology τ on X then $\theta \in \tau$. Hence $\{a, b\} \cup \{b, c\} \in \tau$ and consequently $\{a, b\} \cap \{b, c\} = \{b\} \in \tau$. Thus this intersection $\{b\}$ must be expressed as a union of members from θ, which is impossible. Thus θ cannot serve as a base for a topology on X. □

Proposition 3.4.2. *Let X be a non void set and B be a collection of subsets of X. The following statements are equivalent: If B is a base for a topology on X, then B satisfies the following properties:*

(i) $\{\bigcup \beta : \beta \in B\} = X$

(ii) If β_1 and β_2 are members of B, then $\beta_1 \cap \beta_2$ is a union of members of B

Proof. Assume that B serves as a base for a topology τ on X. Then $B \in \tau$, so every $G \in \tau$ is a union of members of B. Now X is a member of τ, thus $X = \{\bigcup \beta : \beta \in B\}$ and this proves (i).

Next, let β_1 and $\beta_2 \in B$. So $\beta_1, \beta_2 \in \tau$ for $B \in \tau$. Therefore, $\beta_1 \cap \beta_2 \in \tau$ and B is a base for τ. Hence, $\beta_1 \cap \beta_2 =$ a union of members of B which proves property (ii) □

Proposition 3.4.3. *Let (X, ρ) be a metric space, for each $p \in X$, $\epsilon > 0$ and let $\mathcal{N}(p : \epsilon)$ be the set $\{x \in X : \rho(x, p) < \epsilon\}$. Let B be a collection of all such sets $\mathcal{N}(p : \epsilon)$. Then B is a base for the metric topology on X.*

Proof. Now $\bigcap\{\beta : \beta \in B\} = \cup\{\mathcal{N}(p : \epsilon) : p \in X, \epsilon > 0\} \subset X$.

Conversely, if $x \in X$ and $\epsilon > 0$, then $x \in \mathcal{N}(x : \epsilon) \subset \cup\{\mathcal{N}(p : \epsilon) : p \in X, \epsilon > 0\}$ $=\cup\{\beta : \beta \in B\} \Rightarrow X \subset \{\beta : \beta \in B\}$. Thus $X = \{\beta : \beta \in B\}$
So the family B satisfies the requirement (i) to qualify to be a base.

To prove(ii):
Take any two members of B, say $\mathcal{N}_1(p : r_1)$ and $\mathcal{N}_2(q : r_2)$ where $p, q \in X$ and $r_1, r_2 > 0$ and its real.
If $\mathcal{N}_1 \cap \mathcal{N}_2 = \Phi$, then this is a void union of members of B. Otherwise let $\mathcal{N}_1 \cap \mathcal{N}_2 = \Phi$ and let $x \in \mathcal{N}_1(p : \epsilon) \cap \mathcal{N}_2(q : r_2)$
So $x \in \mathcal{N}_1$ and $x \in \mathcal{N}_2$. Therefore, $\rho(x : p) < r_1$ and $\rho(x : q) < r_2 \Rightarrow r_1 - \rho(x : p) = \delta_1 > 0$ and $r_2 - \rho(x : q) = \delta_2 > 0$.

Let $\delta = \min[\delta_1, \delta_2] \Rightarrow \delta > 0$, we claim that $\mathcal{N}(x : \delta) \subset \mathcal{N}_1(p : r_1), \mathcal{N}_2(q : r_2)$
We first prove that $\mathcal{N}(x : \delta) \subset \mathcal{N}_1(p : r_1)$. Let $y \in \mathcal{N}(x : \delta)$ that is, $\rho(y : x) < \delta$

$\therefore \rho(y : p) \leqslant \rho(y : x) + \rho(x : p)$[Triangle inequality in a metric space]

$< \rho + \rho(x : p)$

$\leqslant +\rho(x : p) = r_1$.

Therefore, $\rho(y : p) < r_1$ ie $y \in \mathcal{N}_1(p : r_1)$. Thus $y \in \mathcal{N}(x : \delta) \Rightarrow y \in \mathcal{N}_1(p : r_1)$ i.e.

$\mathcal{N}(x:\delta) \subset \mathcal{N}_1(p:r_1)$.

We can similarly show that $\mathcal{N}(x:\delta) \subset \mathcal{N}_2(q:r_2)$. Thus $\mathcal{N}_1 \cap \mathcal{N}_2$ is a union of members of B which proves part (2) of(ii). Thus B is a base for a topology τ on X. $\square$

Definition 3.4.2. *Let (X,τ) be a topological space. A family $S \subset \tau$ is called a* ***sub-base*** *for the topology τ if the collection of all finite intersection of members of S forms a base for τ*

Thus every open set $G \in \tau$ is a union of a finite intersection of members of S ie

$$G = \cup(S_1 \cap S_2 \cap .... \cap S_n)$$

Proposition 3.4.4. *Any nonvoid collection of subsets of a nonvoid set X serves as a sub-base for a unique topology called the topology generated by a. This topology is also the intersection of all the topologies on X that contain a.*

Proof. Let B be the collection of all finite intersections of members of a. We show that β is a base for a topology on X.

Consider the void intersection of members of a. The latter is X thus $X \in \beta$ and $\Phi \in \beta$.

$$\therefore \cup\{B : B \in \beta\} = X.$$

Next, let B_1 and $B_2 \in \beta$, then by definition of β

$$B_1 = S_1 \cap ... \cap S_{n_i} : S_i \in a$$

$$B_2 = S_1 \cap ... \cap S_{n_i} : S_i \in a$$

Thus $B_1 \cap B_2 = (S_1 \cap ... \cap S_n) \cap (S_1' \cap .... \cap S_n') \in \beta$

$\therefore \beta$ is base for a topology on x. This topology is generated by a and a is a sub base for τ.

Next, denote this topology by $\tau(a)$. We need to show that $\tau(a)$ is the intersection of all topologies on X which contains the given collection a.

Let $\tau_\alpha : \alpha \in \Omega$ be the collection of all topologies on X containing a. Clearly this collection is non void for $p(x)$, the discrete topology contains a.

Now $\cap\tau_\alpha$ is a topology on X which contains a. So $\tau(a)$ is a topology containing a.

$$\therefore \tau(a) \supseteq \cap\tau_\alpha : \alpha \in \Omega \qquad (i)$$

To show the reverse inclusion : $\tau(a) \subseteq \cap\tau - \alpha : \alpha \in \Omega$ $\qquad (ii)$

Let $G \in \tau(a) \Rightarrow G = i \cup (S_i \cap S_i') : S_i, S_i' \in a \subset \tau_\alpha \forall \alpha \in \Omega$

Since $S_i \cap S_i' \in \tau_\alpha \Rightarrow \cup(S_i \cap S_i') \in \tau_\alpha$ i.e. $G \in \cap\tau_\alpha : \alpha \in \Omega$

Thus $G \in \tau(a) \Rightarrow G \in \tau_\alpha : \alpha \in \Omega$ which proves (ii). Hence, $\tau(a) = \cap\{\tau_\alpha : \tau_\alpha \subseteq a\}$ □

Example 3.4.1. *Let $X = \{a, b, c, d\}, Z = \{\{a\}\{a, b\}, \{b, c\}\}$. Find the topology generated by Z*

Proof. First we find the set of all finite intersections of members of a. This gives the basis:

$$\{X, \{a\}, \{a, b\}, \{b, c\}, \{b\}\} = \beta.$$

Now, taking arbitrary union of members of β. This gives the topology generated by Z , that is $\tau(Z)$.

$$\tau(a) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$$ □

3.5 Minimal Generative Bases and Sub-bases of first Hausdorff countable Topological Spaces

We introduce the notion of minimal derivative local bases and use them to characterize some classes of quotient images a kin to first Hausdorff countable topological spaces.

Example 3.5.1. *A topological space (X, τ) on a base B which induces reducible conditions on general space, is called a minimal generative base.*

Consider the well known real line $\mathbb{R}$ induced with a set $B = \{(x, -x) : x \in \mathbb{R}\}$. Here we can consider arbitrary open sets, obtained from B with: $B = \{(x, -x) \cup \Phi :$

$x \in \mathbb{R}\}$ and $\bigcup G_x = B$:

$$x = \begin{cases} \frac{1}{n} \; 0 < x < 1; \\ \\ n \; a > 1 \end{cases}$$

$$-x = \begin{cases} \frac{-1}{n} \; 1 < x < 0; \\ \\ -n \; x < 1 \end{cases}$$

We can associate the set B above with a topology $T = \{\Phi, (x, -x)\}$

Lemma 3.5.1. *Let X be a Quotient image space and B_M a family of open sets. Then we say that B_M is the minimal generative base for the topology of X if:*

i B_M covers X

ii If $A, B \in B_M$ then there exists a subfamily $\{B_i : i \in \mathbb{N}\}$ of B_M such that $A \cap B = \bigcup_{i \in \mathbb{N}} B_i$

iii If a subfamily $\{B_{i \in \mathbb{N}} \mid i \in N\} \subset B_M$ substantiates $\bigcup_{i \in \mathbb{N}} B_i \in B_M$ then, there exist $i_\circ \in \mathbb{N}$ such that $\bigcup_{i \in \mathbb{N}} B_i = B_{i_\circ}$

Example 3.5.2. . *Let $B_M = \{(-x, x) \cup \Phi; \forall x \in \mathbb{R}\}$ and $\tau = \{(x, -x), \Phi, \mathbb{R}\}$*

Then $\bigcup B_M = B_M \cup \tau = \mathbb{R}$. For any subsets $A, B \in B_M; A \subset B$ or $B \subset A$. Now assume that $A \subset B$. Then there exists a subfamily $\{B_i : B_i \subset A, i \in \mathbb{N}\}$ such A is a refinement of $A \subset B$ so that $A \cap B = A = \bigcup\{B_i : B_i \subset A, i \in \mathbb{N}\}$. If this sub family qualifies B_M to be a base, then $B_M = \bigcup\{B_i : i \in \mathbb{N}\}$ because clearly $\bigcup\{B_i : i \in \mathbb{N} \in B_M$ and $B_M \subset \bigcup\{B_i : i \in \mathbb{N}\}$. Thus the base B_M satisfies the conditions of the previous lemma 3.5.1)

In the next result, we demonstrate the hereditary condition for usual bases to be minimal generative

Theorem 3.5.1. *Let X be a set and B_M a family of non-empty subsets of X. Then B_M is the minimal generative base for a topology of X if and only if:*

i B_M *covers* X*; that is* $X = \bigcup_{M \in \mathbb{N}} B_M$

ii If $A, B \in B_M$*, then there exists a subfamily* $\{B_i : i \in \mathbb{N}\}$ *of* B_M *such that* $A \cap B = \bigcup_{i \in \mathbb{N}} B_i$.

iii If a subfamily $\{B_i : i \in I\}$ *of* B_M *satisfies* $\bigcup_{i \in \mathbb{B}} B_i$ *then there exists* $i_o \in \mathbb{N}$ *such that* $\bigcup_{i \in \mathbb{N}} B_i = B_{i_0}$

Proof. Suppose B_M is a minimal generative base for a topology in the space X. Then the conditions (i) and condition (ii) of 3.5.1 have to hold. Assume a subfamily $B_{M_1} = \{B_i : i \in \mathbb{N}\}$ of B_M satisfies $\bigcup_{i \in \mathbb{N}} B_i \in B_M$ but $\bigcup_{i \in \mathbb{N}} B_i \neq B$ for each $B \subset B_M$. Therefore there exists a subset $B' \in B_M \setminus B_i$ such that $\bigcup_{i \in \mathbb{N}} B_i = B'$. That implies that $B_M \in B'$ is a base for the topological space X. This is a contradiction and thus condition (iii) of 3.5.1 holds. conditions 3.5.1 and 3.5.1 states that B_M is a base for a topological space X which is true from above. If a subfamily B'_M of B_M is a base of the topological space X, then for every subset $B \in B_M$, there exists a subfamily $\{B_i : i \in \mathbb{N}\}$ ofB'_M such that $\bigcup_{i \in \mathbb{N}} B_i = B_{i_0}$. In relation to condition 3.5.1, there exists $i_0 \in \mathbb{N}$ such that $\bigcup_{i \in \mathbb{N}} B_i = B_{i_0}$. Hence $B = B_{i_0} \in B'_M = B_M$. Therefore, B_M is minimal generative base for topological space X. □

Definition 3.5.1. *Let* X *be a topological space and* B_M *a family of non-empty subsets of* X*. Then for each subset* $B \in B_M$.

i if there exists a subfamily $\{B_i : i \in \mathbb{N}\}$ *of* B_M *such that* $B \notin \{B_i : i \in \mathbb{N}\}$ *and* $\bigcup_{i \in \mathbb{N}} B_i = B$ *then* B *is known as a union reducible element with respect to* B_M*;*

ii If there exists a subfamily $B_i : i \in \mathbb{N}$ *of* B_M *such that* $B \notin \{B_i : i \in \mathbb{N}\}$ *and,* $\bigcap_{i \in \mathbb{N}} B_i = B$ *then* B *is known as an intersection reducible element with respect to* B.

Remark 3.5.1. *A minimal generative local base for a first continuous topological space may fail to have a union reducible candidate and still poses the intersection reducible elements. This property has been applied before. See for example in [40] to qualify the minimality of an arbitrary base say B_N with a well known countable set which is a topological space with respect to a respected topology τ. Indeed the classification into reducibility using sub-bases usually defined over the intersections of union of bases all over for a consideration of only intersection reducibility for local minimal generative bases considered for the quotient of first Hausdorff countable topological spaces*

Next we characterize the neccessitors and sufficients for minimal generative sub-minimals of B_M called sub-basis

Definition 3.5.2. *Let S be a sub-base for a topological space X for any subfamily S' of S. If S' is a sub-base for the topological space X, then $S' = S$.*

The following example helps us to illustrate the definition 3.5.2 ;

Example 3.5.3. *Let X $=\mathbb{R} \setminus Z$ and $S = \{(a, a+1) : a \in \mathbb{Z}\}$. Then, S is a partition of $\mathbb{R} \setminus \mathbb{Z}$. This therefore implies that S is a minimal generative sub-base for a topology of $\mathbb{R} \setminus \mathbb{Z}$.*

Proposition 3.5.1. *Let X be a set and S a family of non-empty subsets of X. If S is a minimal generative sub-base for a topology of X, then there doesn't exist a finite intersection reducible element in the minimal generative sub-base S.*

Proof. By definition 3.5.1 above, we need to show that if a finite subfamily $\{S_i : i \in \mathbb{N}\}$of S satisfies $\bigcap_{i\in\mathbb{N}} S_i \in S$, then there exists $i_0 \in \mathbb{N}$ such that $\bigcap_{i\in\mathbb{N}} S_i =S_{i_0}$. Now assume that a finite subfamily $S = \{S_i \mid i \in \mathbb{N}\}$ of S satisfies $\bigcap_{i\in\mathbb{N}} S_i \in S$, but $\bigcap_{i\in\mathbb{N}} S_i \in I \neq S$ then for each subset $S \in S$. Therefore there exists a subset $S\prime \in S \setminus S_1$such that $\bigcap_{i\in\mathbb{N}} S_i = S\prime$. Thus this implies that $S \in \{S\}\prime$ is a sub-base for the topology of X of which is a contradiction. Therefore, the minimal generative sub-base S doesn't contain the finite intersection reducible elements.

From the following proposition, it is evident that a minimal generative sub-base is a minimal generative base with all finite intersection reducible elements removed. according to the following proposition. □

Proposition 3.5.2. *Let X be a topological space and B_M be a minimal base for the Topological Space. If there are no finite intersection reducible elements in B_M, then B_M is a minimal generative sub-base for the Topological Space X.*

Proof. Clearly, B_M can be viewed as a sub-base for the topological space$X$. Suppose a subfamily l of B_M is a sub-base for the topological space X. According to the definition of base and sub-base, B_M is a sub-base, B_M is a minimal base generated by S_M. Then for each subset $B \in B_M$, there exists a finite subfamily $\{S_i i \in I\}$ of S_M such that $\bigcap_{i\in\mathbb{N}} S_i = B \in B_M$. There exists $i_0 \in I$ such that $\bigcap_{i\in\mathbb{N}} S_i = S_{i_0}$, because the minimal base B_M does not have finite intersection reducible elements. Thus, $B = S_{i_0} \in S_M$, which means $S_M = B_M$. Therefore, B_M is a minimal generative sub-base for the topological space X

□

Proposition 3.5.3. *Suppose X is a set and is a Y topological space. Then if $f : X \to Y$ is a surjection map and S_M is a minimal generative sub-base for the topological space Y, the $f^{-1}(S_M) = \{f^{-1}(S) : S \in S_M\}$. is a minimal sub-base for a topology on X*

Proof. Given that S_M is a minimal generative sub-base of the topological space Y, then $Y = \bigcup S_M$ and we $X = \bigcup f^{-1}(S_M)$. Hence, $f^{-1}(S_M)$ is a sub-base for the topology τ on X. Suppose $f^{-1}(S_M)$ is not a minimal generative sub-base for the topological space (X, τ), then there exists a $S_0 \in S_M$ such that $f^{-1}(S_M) \setminus f^{-1}(S_0$

is a sub-base for the topological space (X, τ). Given that $S_M \setminus (S_0)$, is not a sub-base for the topological space Y, then there exists an open subset V of Y and $y \in V$ such that for any finite family $F_M \subset S_M \setminus (S_0), y \bigcap F_M \subset V$ does not hold. Take $x \in f^{-1}(y)$. Because $f^{-1}(V)$ is open in the topological space (X, τ) and $f^{-1}(S_M)\{f^{-1}(S_0)$ is a sub-base for (X, τ), there exists a finite family $F_M \subset S_M \setminus S_0$ such that $x \in \bigcap\{f^{-1}(F) : F \in F_M\} \subset f^{-1}(V)$. Hence, $y \in \bigcap\{F : F \in F_M\} = \bigcap F_M \subset V$. Thus, $f^{-1}(S_M)$ is a minimal generative sub-base for a topology on X □

Based on proposition 3.5.3, we obtain the following remark.

Remark 3.5.2. *Let Y be a quotient space of a topological spaces X and let $f : X \to Y$ be a quotient mapping. If a covering S_M is a minimal generative sub-base for the quotient space Y, then $f^{-1}(S_M) = \{f^{-1}(S) : S \in S_M\}$ is a minimal generative sub-base for the topological space X*

Proposition 3.5.4. *Let X and Y be topological spaces and a mapping be $f : X \to Y$ open and 1-1. If a covering S_M is a minimal generative sub-base for the topological space X, then $f(S_M) = \{f(S) : S \in S_M\}$ is a minimal generative sub-base for the subspace $f(X)$ of Y*

Proof. It's obvious that, $f(S_M) = \{f(S) : S \in S_M\}$ is a cover of $f(X)$. Because the mapping $f : X \to Y$ is open and a sub-base for the topological space $X, f(F_M)$ is a sub-base for the topological space $f(X)$. We now show that, $f(S_M)$ is minimal generative base. Now lets suppose that $f(S_M)$ not minimal generative sub-base for $f(X)$. Therefore there exists $S_0 \in S_M$ such that $\{f(S) : S \in S_M \setminus \{S_0\}\}$ is a sub-base for $f(X)$. Since S_M is a minimal generative sub-base for the topological space $X, S_M \setminus \{S_0\}$ is a sub-base for the topological space X. Therefore, there exists an open subset V_0 of X and $x \in V_0$ such that for any finite family $F_M \subset S_M \setminus \{S_0\}, \bigcap F_M \subseteq V_0$ if $x \in \bigcap F_M$. Now since the mapping is open, $f(V_0)$ is an open subset of $f(X)$. And since f is 1-1, $f(\bigcap F_M) \subseteq f(V_0)$ for any finite family $F_M \subset S_M \setminus S_0$ with $x \in \bigcap F_M$ Thus, $\bigcap\{f(F) : F \in F_M\} \subseteq f(V_0)$ for any finite family $F_M \subset S_M \setminus \{S_0\}$ with $x \in \bigcap F_M$. This is a contradiction that $f(S_M) \setminus \{f(S_0)\}$ is a sub-base for $f(x)$ $\square$

Next some propositions have been presented as mapping properties.

Proposition 3.5.5. *Let $\{X_y\} : y \in \Gamma$ be a family of topological spaces. Then for each index such that $y \in \Gamma, \exists \pi_y : \pi_{y\in\Gamma} X \to X_y$ is a projective mapping and for each index $y \in \Gamma, S_{M_y}$ in that a minimal generative sub-base for topological space X, then $S_M = \{\pi_y^{-1}(S_y) : S_y \subset S_{M_y} \in \Gamma\}$ is a minimal generative sub-base for the product space $\pi_{y\in\Gamma} X_y$.*

Proof. It is clear that $S_M = \{\pi_y^{-1}(S_y) : S_y \subset S_{M_y} \in \Gamma\}$ is a sub-base for the product space $\prod_{y\in\Gamma} X_y$. Suppose S_M is not a sub-base of the product space $\pi_{y\in\Gamma} X_y$, there exist a $y_0 \in \Gamma$ and $S_{y_0} \in S_{M_{y_0}}$ such that $S_M \setminus \{\pi^{-1 y_0}(S_0)\}$ is a sub-base of the product space $\prod_{y\in\Gamma} X_y$. Hence, $S_{M_{y_0}} \setminus \{S_{y_0}\}$ is a sub-base of the space X_{y_0}. This is a contradiction that $S_{M_{y_0}}$ is a minimal generative sub-base for X_{y_0} $\square$

Proposition 3.5.6. *If $\{X_y\} : y \in \Gamma$ be a family of pairwise disjoint topological spaces. For each index $y \in \Gamma$, if S_{M_y} is a minimal generative sub-base for a topological space (X_y, τ_y), then $S_M = \{S_y : S_y \in S_{M_y}, y \in \Gamma\}$ is a minimal generative sub-base for the sum of the spaces $\{X_y\}_{y\Gamma}\}$*

Proof. If $(\oplus, X_y)$ is the sum of the spaces $\{X_y\}_{y\in\Gamma}$, let $B_{M_y} = \{\bigcap_{S\in\Gamma} S_y : S'_{M_y}\}$ that is a finite subfamily of S_{M_y} denote the base generated by S_{M_1}. It can be seen that $B_M = \{B_y : B_y \in B_{M_{y,y\in\Gamma}}\}$ is a base generated by S_M. For each open set $U \in \tau$ and each point $x \in U$, then there exists an index $y \in \Gamma$ such that $x \in X$. This therefore follows that $x \in U \cap X_y$ and there exists a subset $\bigcap S_{y\in S_{M_y}} S_y \in B_{M_y} \subset B_M$ such that $x \in \bigcap S_y$. Then $x \in S_{y\in S_M} S_y \subset U \cap X_y$. Therefore there exists a subset $\bigcap S_{y \in S_{M_y}} S_y \subset B_M$ such that $x \in \bigcap S_{y \in S_{M_y}} S_y \in B_{M_y} \subset B_M$ such that $x \in \bigcap S_{y \in S_{M_y}} S_y \subset U \cap X_y$ is open in $X_y, x \in \bigcap_{S_{y \in S_{M_y}}} S_y \subset U$ meaning that B_M is a base for the topological space $(\oplus_{y \in \Gamma}, X_y, \tau)$. Thus, S_M is a sub-base for the topological space $(\oplus_{y \in \Gamma}, X_y, \tau)$ Now suppose S_M is not a minimal generative sub-base for the topological space $\oplus_{y \in \Gamma} X_y, \tau$. Then there exists $y_0 \in \Gamma$ and $S_{y0} \in S_{M_{y0}}$ such that $S_M \backslash (S_{y0})$ is a sub-base of the topological space $\oplus_{y \in \Gamma} X_y, \tau$. Hence, $S_{M_y} \setminus \{S_y\}$ is a sub-base of X_{y_o}. This is a contradiction and hence S_M is a minimal generative sub-base for the Topological Space $\oplus_{y \in \Gamma} X_y, \tau$.

Let X be a Topological Space and Y be a subset of X which is a Topological subspace of X. If S_M is a minimal generative sub-base for the topological space X, then $S_M - y$ is a sub-base for the topological subspace Y. But $S_M - y$ may not be a minimal generative sub-base for the subspace. The following example is used to illustrate this point.

Example 3.5.4. *Let $a \in X$ and $X = \{a_1, a_2, \cdots a_9\}$*

$$S_M = \{\{a_1, a_2, a_4, a_5\}, \{a_2, a_3, a_5, a_6\}, \{a_4, a_5, a_7, a_8\}, \{a_5, a_6, a_8, a_9\}\}$$

be a sub-base for a topological space X. It is easy to see that S_M is a minimal generative sub-base for the topological space X. Take $Y = \{a_1, a_2, a_4, a_5, a_7\}$. According to the definition of a topological sub-space, we conclude that $S_M - y = \{\{a_1, a_2, a_4, a_5\}, \{a_2, a_5\}, \{a_4, a_5, a_7\}, \{a_5\}\}$ is a sub-base for the subspace Y. However, $S_M - y$ is not a minimal generative sub-base for the subspace Y, because $\{a_5\}$ is a finite intersection reducible elements with respect to S_{M_y}.

□

3.6 Some Special Results in First Hausdorff Countable Hausdorff Spaces

We call a topological space X a first Hausdorff countable Hausdorff space if and only if each point in X has a unique minimal open neighborhood. Moreover, for the first Hausdorff countable Hausdorff space there exist a base B_M such that B_M is composed of all the minimal open neighborhoods of each point in X. With no doubt B_M is the unique minimal base for the quotient space X. By proposition 3.5.2 a minimal generative sub-base can be obtained from a minimal generative base.

Proposition 3.6.1. *For any first Hausdorff countable Hausdorff space, the minimal generative sub-base is not unique We can illustrate this the following using example;*

Example 3.6.1. *Consider* $X = \{a_1, a_2, a_3, a_4\}$

$$S_M = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_1, a_4\}, \{a_2, a_4\}, \{a_1, a_3\}\}$$

be a sub-base for the Topological Space $X(\tau)$.

From the topological space (X, τ), *the following Minimal generative sub-bases can be derived.*

$$F_M S_1 = \{\{a_1, a_2\}, \{a_3, a_4\}, \{a_2, a_4\}, \{a_1, a_3\}\}$$
$$F_M S_2 = \{\{a_2, a_3\}, \{a_1, a_4\}, \{a_3, a_4\}, \{a_1, a_4\}\}$$
$$F_M S_3 = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_1, a_4\}\}$$

Remark 3.6.1. *Let* X *be a topological space. If* B_M *is a base for the topological space* X *and* S_M *is a sub-base for the topological space* X. $N_\tau(x) = N_{B_M}(x) N_{S_M(X)}$ *can be derived according to the definitions of base and sub-bases. In addition, if* (X, τ) *is a first Hausdorff countable Hausdorff space, then* $N_\tau(x) = \bigcap\{U \mid x \in U \in \tau\}$ *is the unique minimal open neighbourhood of each point* $x \in X$. . *Now denote*

$N_\tau(x)$ by $N(x)$ and let $S_M = \{N(X) \mid x \in X\}$ which is the unique minimal generative base for the first Hausdorff countable Hausdorff space being a partition
By our previous example, there exists an intersection reducible elements in a minimal generative base. As a special case, if a minimal generative base is a partition, then there is no any intersection reducible element. Hence when a minimal generative base is a partition, then there is a discrete topological space. To help us get a better understanding of a minimal generative base of first Countable space that is a partition a proposition has been provided.

Proposition 3.6.2. *Let X be a first countable Hausdorff space. For two points $x, y \in X$, define xRy if $N(X)$ =$N(y)$. Then the natural quotient space $X \setminus R$ is a discrete space if and only if the minimal generative base B_M is a partition for the first Hausdorff countable Hausdorff space X*

Proof. Suppose $X \setminus R$ be quotient mapping that is also natural and Suppose $X \setminus R$ is a discrete space, then for each point $x \in X$, $\{[x]\}$ is a singleton in $X \setminus R$ and $p^{-1}([x])$ is open in X. Thus $N(X \subset p^1([x])$. For each point $y \in p^{-1}([x]), xRy$ shows that $N(x) = N(y)$ that is $y \in N(x)$. Then $p^{-1}([x]) \subset N(x)$, so $p^{-1}([x]) = N(x)$. Thus for any two minimal open neighbourhoods $N(x), N(y) \in B_M$ and $N(x) \neq N(y)$ if $N(x) \cap N(y) \neq 0$, then there exists a point $a \in X$ such that $a \in N(x) \cap N(y)$. That is $a \in N(x)$ and $a \in N(y)$. So $a \in p^{-1}([x])$ and $a \in p^{-1}([y])$ that is xRaans yRa. Then xRy because R is an equivalence relation, which means $N(x) = N(y)$. Therefore, for any elements $N(x), N(y) \in B_M$, we have that $N(x) = N(y)$ or $N(x) \cap N(y) = \varnothing$ that is ,B_M is a partition
Because B_M is a partition, $N(x)$= $N(y)$ if $N(X) \cap N(y) = \varnothing$. Then for each point $y \in N(x), N(X) = N(y)$, that is xRy. Thus, $y \in p^{-1}([x])$ which implies that $N(x) \subset p^{-1}([x])$. If $yinp^{-1}([x])$, then xRy which means $N(X) = N(y)$. So $p^{-1}([x]) \subset N(X)$. Therefore, $p^{-1}([x])$= $N(x)$ is open in X. Since p is a quotient mapping that is also natural, then $\{[x]\}$ is open in $X \setminus R$. Hence $X \setminus R$ is a discrete

space □

Next we present results about locally connected spaces and locally pathwise connected spaces

Proposition 3.6.3. *Let X be a first Hausdorff countable Hausdorff space:*

i A space X is said to be locally connected space if and only if each point $x \in X$, there is a minimal open neighborhood of the point x that is a connected set

ii A space X is said to be locally pathwise connected space if and only if for each point $x \in X$, then the minimal open neighborhood of the point x is also pathwise connected set

Proof. (i) Let X be a locally connected space. Then for each neighbourhood U of x such that $x \in X$, there exists a connected neighborhood V such that $x \in V \subset U$. This implies that the minimal open neighborhood of the point x is a connected set. Let $x \in X$, then the minimal open neighborhood $N(x)$ of the point x is a connected set. For each neighborhood U of the point $x, N(x) \subset U$. Thus, X is a locally connected space.

The proof for (ii) is similar to (i)

For our first countable Hausdorff space, based on the uniqueness of minimal base, two results about sub-base and minimal generative sub-base are presented.

□

Lemma 3.6.1. *Let τ, τ' be two topologies on X such that X is a first Hausdorff countable Hausdorff space and $N_\tau(x)= N_{\tau\prime}(x)$ for all $x \in X$, then $\tau = \tau\prime$*

Proof. Since X is first-countable, each point $x \in X$ has a countable neighborhood basis. Given that $N_\tau(x)= N_{\tau\prime}(x)$ for all $x \in X$, every neighborhood of $x \in \tau$ is also a neighborhood in $\tau = \tau\prime$ and vice versa. Therefore, the collection of open sets generated by these neighborhoods will be identical in both topologies, which implies $\tau = \tau'$ □

Proposition 3.6.4. *Let (X, τ) be first countable Hausdorff space. A covering S_M is a sub-base for the first Hausdorff countable Hausdorff space if the following conditions hold:*

i S_M is a sub-base for a topological space X, each point in x has the minimal open neighborhood.

ii $N_{S_M}(X)= N(x)$ for each Point $x \in X$

Proof. Suppose S_M is a sub-base for a topological space $(X, \tau\prime$ and $(X, \tau\prime)$ is a first countable space, since each point x in the topological space $(X, \tau\prime)$ has the minimal open neighborhood, then $N_{S_M}(x)$ for each point $x \in X$ and $(X\tau)$ is first countable Hausdorff space, by lemma 3.6.1, $\tau = \tau\prime$. Hence, S_M is a sub-base for the first countable Hausdorff space (X, τ).

Suppose S_M is a generative sub-base for the first Hausdorff countable Hausdorff space (X, τ). Then each point in x in the first Hausdorff countable Hausdorff space (X, τ) has the minimal open neighbourhood. Hence for each point $x \in X$, $N_{S_M}(X) = N(x)$ for each point $x \in X$

□

Proposition 3.6.5. *Let S_M be sub-base for first countable Hausdorff space (X, τ). S_M is a minimal generative sub-base for the first Hausdorff countable Hausdorff space (X, τ) if and only if for any covering $S'_M \subset S_M$, if and only if the following conditions hold, then $S_M = S_M\prime$.*

i If S'_M is a sub-base for a topological space X, then each point in x in the topological space X has the minimal open neighbourhood

ii $N_{S'_M}(x) = N(x)$ for each point $x \in X$

Proof. Let a covering $S_{M\prime} \subset S_M$ be a sub-base for the first countable Hausdorff space $(X \in \tau)$. Then by proposition 3.6.3, conditions (i) and (ii) hold. Thus $S_{M\prime}= S_M$. Hence S_M is a minimal generative sub-base for (X, τ). Now suppose S_M is a minimal generative sub-base for the first countable Hausdorff space $(X, \tau,)$ then for any covering $S_{M\prime} \subset S_M$, if Conditions (i) and (ii) hold, by proposition 3.6.3 then S_M is a sub-base for the first countable Hausdorff space (X, τ). Thus, $S'_M = S_M$ since S_M is a minimal generative sub-base for the first countable Hausdorff space (X, τ). □

Proposition 3.6.6. *Let X be a first countable Hausdorff space and S_M be a family of open sets of X. Then S_M is a minimal generative sub-base for the topology of X if and only if:*

i S_M is a sub-base for the first countable Hausdorff space X;

ii If a subfamily $\{S_i \mid i \in I\}$of S_M satisfies $\bigcap_{i \in I} S_i \in S_M$,then there exists $i \in I$ such that $\bigcap_{i \in I} S_i = S_{i_0}$

Proof. Let S_M be a minimal generative sub-base for the first countable Hausdorff space. Suppose we assume a subfamily $S_{M_1} = \{S_i \mid i \in I\}$ of S_M satisfies $\bigcap_{i \in I} S_i \in S_M$, but $\bigcap_{i \in I} S_i \notin S_{M_1}$ i.e $\cap_{i \in I} S_i \in S_M \backslash S_{M_1}$. Then there exists a subset $S' \in S_M \backslash S_{M_1}$ such that $\cap_{i \in I} S_i = S'$. Then for each point $x \in S'$,

$$
\begin{aligned}
N_{S_M} &= \cap \left\{ S \in S_M \middle| x \in S \right\} \\
&= \left(\cap \left\{ S \in S_M \backslash S' \middle| x \in S \right\} \right) \cap S' \\
&= (\cap \{ S \in S_M \backslash S' \Big| x \in S \}) \cap (\cap_{i \in I} S_i) \\
&= \cap \left\{ S \in S_M \backslash S' \middle| x \in S \right\} \\
&= N_{S_M \backslash \{S'\}}(x)
\end{aligned}
$$

Thus for each point $x \in X$, $N_{S_M}(x) = N_{S_M \backslash \{S'\}}(x)$. We can also say that, $S_M \backslash \{S'\}$

is a sub-base for the first countable Hausdorff space (X, τ'). According to proposition 3.6.3, $S_M \backslash S'$ is a sub-base for the first countable Hausdorff space (X, t) which is a contradiction. Thus conditions in proposition 3.7.5 holds.

We now prove that a generative sub-base S_M is minimal. Let S'_M of S_M be a subfamily of a sub-base for the first countable Hausdorff space (X, τ), but $S'_M \neq S_M$. Then there exists an open set $S' \in S_M$ such that $S' \notin S'_M$. According to proposition 3.7.5, for any subfamily $\{S_i \Big| i \in I\}$ of $S_M \backslash S'$, $\cap_i \in IS_i \neq S'$. Thus, there exist a point $x \in S'$ such that $N_{S'_M}(x) \neq N_{S_M}(x)$. This is a contradiction and thus $S'_M = S_M$. By the definition of minimal generative sub-base, S_M is a minimal generative sub-base for the first countable Hausdorff space (X, τ). □

Proposition 3.6.7. *Suppose S_M is a sub-base for a Topological Space (X, τ) and Y a Topological Space, for two points $x_1, x_2 \in X$, lets define $x_1 R x_2 = NS_M(x_2$. A mapping $\rho : X \to X/R$ is a natural quotient mapping. Now suppose a mapping $g : X/R \to Y$ is a bijection. If a mapping $f : X \to Y$ satisfies $f = g \circ p$, then the following holds:*

- *i For each subsets $S \in S_M, x_1 \in S$ implies $x_2 \in S$ for any points $x_1, x_2 \in X$ satisfying $f(x_1) = f(x_2$*
- *ii For any subset $S_1, S_2 \in S_M, f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$;*
- *iii $f(NS_M(x)) = \cap\{f(S) : x \in S \in S_M\}$ for each point $x \in X$;*
- *iv For each subset $S \in S_M, f^{-1}(f(S)) = S$;*
- *v $f^{-1}(f(NS_M(x)) = NS_M(x)$ for each point $x \in X$.*

Proof. (i) We first prove that $f(x_1) = f(x_2$ means that $NS_M(x_1) = NS_M(x_2)$ for any points $x_1, x_2 \in X$ because $f = g \circ p$ and $f(x_1) = f(x_2)$ implying that $g(p(x_1)) = g \circ p(x_1) = g \circ p(x_2)$. Therefore $p(x_1) = p(x_2)$ means that

$g(p(x_1)) = g \circ p(x_1) = g(p(x_2))$. Finally, for each subset $S \in S_M, x_1 \in S$ implies $x_1 \in S_M(x_1 \subset S$. $NS_M(x_1) = NS_M(x_2)$ since $f(x_1) = f(x_2)$. This implies that $NS_M(x_1) \subset S$. That is $x_2 \in S$.

(ii) We prove that $f(S_1) \cap f(S_2) = \emptyset$ if $S_1 \cap S_2 = \emptyset$. Assume by contradiction that $f(S_1) \cap f(S_2) \neq \emptyset$. There then exists a point $y \in Y$ such that $f(S_1) \cap f(S_2)$ that is $y \in f(S_1)$ and $y \in f(S_2)$. Let there exists two points $x_1 \in S_1$ and $x_2 \in S_2$ such that $f(x_1) = f(x_2) = y$. By (i), $x_2 \in S_1$. Then $x_2 \in S_1 \cap S_2$. Which is a contradiction. Thus, $f(S_1) \cap f(S_2 = \emptyset$. Next, we show that if $S_1 \cap S_2 \neq \emptyset$, then $f(S_1 \cap S_2) = f(S_1) \cap f(S_2$. It's obvious that, $f(S_1 \cap S_2) \subset f(S_1) \cap f(S_2)$. For each point $y \in f(S_1) \cap f(S_2)$, there exists two points $x_1 \in S_1$ and $x_2 \in S_2$ such that $f(x_1)$=$f(x_2)$=y. By (3.6.7), $x_2 \in S_1$. Then $x_2 \in S_1 \cap S_2$, which means $y = f(x_2) \in f(S_1 \cap S_2)$. So $f(S_1) \cap f(S_2) \subset f(S_1 \cap S_2)$. That is $f(S_1 \cap S_2)$=$f(S_1) \cap f(S_2)$

(iii) For (iii)the proof is similar to (ii) .

(iv) It is clear that $S \subset f^{-1}(f(S))$ is always true. For each point $x \in f^{-1}(f(S) \subset S$. That is $f^{-1}(f(S)) = S$.

(v) The proof for (v) is similar to (iv).

□

Lemma 3.6.2. *Let X be a Topological Space and R an equivalence relation. If the mapping $p : X \to X/R$ is assumed to be a natural quotient mapping, then a mapping g of a quotient space X/R to a Topological Space Y is continuous if and only if the composition $g \circ p$ is continuous.*

Proof. Assume $p : X/R \to Y$ is continuous. Since p is a quotient map, the continuity of $g \circ p$ follows directly because the composition of two continuous functions is continuous.Conversely, suppose $g \circ p$ is continuous. For any open set $V \subset Y$,

$(g \circ p)^{-1}(V)$is open in X by the continuity of $g \circ p$. Because p is a quotient mapping, an open subset $W \subset X/R$ is open if and only if $P^{-1}(W)$ is open in X. Since $(g \circ p)^{-1}(V) = p^{-1}(g^{-1}(V))$ is open in $X, g^{-1}(V)$ must be open in X, g^{-1} must be open in X/R, implying g is continuous $\square$

Proposition 3.6.8. *Let S_M be a sub-base for first countable Hausdorff space (X, τ_x) and Y a Topological space. For two points $x_1, x_2 \in X$, define $x_1 R x_2$ if $NS_M(x_1) = NS_M(x_2)$. Let a mapping be defined as $p : X \to X/R$ a natural quotient mapping. Let a mapping $g : X/R \to Y$ be a continuous bijection. If a mapping $f : X \to Y$ is an open mapping satisfying $f = g \circ p$ and S_M is a minimal generative sub-base for the first countable Hausdorff space (X, τ_x), then $f(S_M)$ is a minimal generative sub-base for first countable Hausdorff space (Y, τ_y).*

Proof. Since g is a continuous bijection, then by lemma 3.6.2, f is an open and continuous surjection and $(Y, \tau y)$ is countable space. It,s clear that, $f(S_M)$ is a covering of Y. By Proposition 3.6.7 (3), $f(NS_M(x)) = \cap\{f(S)|x \in S \in S_M\} = \cap f(S) \in f(S_M) = N_(S_M)(f(x))$ for each point $x \in X$. Because S_M is a minimal sub-base for the first countable Hausdorff space, then, by proposition 3.6, $N_{S_M}(x) = N(x)$ for each point $x \in X$ and $NS_M(x)$ is open in $(X, \tau x)$. Then $N_{f(S_M)}(f(x)) = f(N_{S_M(x)})$ is open in $(Y, \tau y)$ because f is an open mapping. It is obvious that $N_{fS_M}(x)$ is the minimal set containing $f(x)$ for each point $x \in X$. Thus $N_{fS_M}(f(x))$ is the minimal open neighborhood of $f(x)$. Also, $N_{f(S_M}(f(x)) = f(N_{S_M(x)} = f(N(x)) = N(f(x))$ for each point $x \in X$. Thus, $f(S_M)$ is a sub-base for the described topological space $(Y, \tau y)$.

For any covering $f(S'_M) \subset f(f)$, suppose each point space Y generated by $f(S'_M)$ has the minimal open neighbourhood and $N_{f(S'_M}(y) = N(y)$ for each $y \in Y$. Then S'_M is a sub-base for the first countable Hausdorff space $(X, \tau x)$. If otherwise, there exist points $x, x' \in X$ such that the minimal open neighborhood of x doesn't exist or $N_{S_M(x')} \neq S_M(x')$. It is easy to see that there exist two points $y, y' \in Y$ such that

$y = f(x)$ and $y' = f(x')$. From these proof, we are able to obtain that the minimal open neighborhood of y does not exist or $N_{f(S_M)(y')} \neq N(y')$. This is a contradiction. So $F_M S'$ is a sub-base for the first countable Hausdorff space $(X, \tau x)$. Because S_M is a minimal generative sub-base for the first countable Hausdorff space $(X, \tau x)$, by definition 3.5.2, $F_M S' = S_M$. Thus, $f(F_M S') = f(S_M)$. By proposition 3.6.3 , a minimal generative sub-base for the first countable space is (Y, τ_y). $\square$

CHAPTER FOUR

ORDINAL RANKS OF HAUSDORFF SPACES

4.1 Introduction

In this chapter, we have determined whether the Cantor-Bendixson rank of the Hausdorff topological space over a real line $\mathbb{R}$ is first uncountable ordinal. We also showed that the specific characteristics of a Hausdorff space has a relation to the Cantor-Bendixson rank.

That is the smallest ordinal that cannot be put into a 1-1 correspondence with any countable set of ordinals for example the natural numbers.

4.2 Ordinal Ranks

Remark 4.2.1. *The **real line** with its standard topology is topological space with a well defined topology. The Cantor-Bendixson rank of $\mathbb{R}$ helps us know how we can repeatedly take limit points and remove any isolated points*

Proposition 4.2.1. *The rank of $(\mathbb{R}, d)$ is a first uncountable ordinal(ω_1).*

Proof. The continuum hypothesis(CB) states that there is no set with a cardinality strictly between the set of natural numbers and the set of real numbers. The continuum hypothesis simply means that the size of the continuum is the smallest uncountable cardinality.

Now to determine whether the Cantor-Bendixson rank of a real line is first uncountable ordinal, we need to consider the real line $\mathbb{R}$, which is endowed by the standard topology. To get the Cantor-Bendixson rank of a real line line $\mathbb{R}$, we start by taking the set of all limit points of $\mathbb{R}$ and we call them the first derivative. As much as in the first derivative we may have removed some of the isolated points, it's clear that there are some of the isolated points left. We therefore repeat the process by taking the limit points of the first derivative and creating the second derivative. This process continues and each step we eliminate some of the isolated points but we still have some left. this process eventually stabilizes and we therefore can longer eliminate any isolated points.

Assuming the continuum hypothesis, implies that there are no sets with a cardinality strictly between the set of natural numbers $\mathbb{N}$ and the set of real line $\mathbb{R}$. As a result, the action of taking the Cantor-Bendixson derivatives stabilizes after finitely many steps. This therefore implies that there is no set of real numbers that requires countably many steps to reach its Cantor-Bendixson rank.

Since the process stabilizes after many steps, the Cantor-Bendixson rank of the real line $\mathbb{R}$ is therefore the least ordinal that represents the point at which the process stabilizes. This is ordinal is what we call the first uncountable ordinal (ω_1). from our knowledge of set theory, (ω_1) has the smallest uncountable cardinality and therefore its used to describe the size and complexity of sets and their structures.

And so when we assume the continuum hypothesis, the Cantor-Bendixson rank of the real line is therefore ω_1 as it represents the ordinal at a point where derivatives are successively eliminating isolated points into stabilizers and $\mathbb{R}$ is fully classified in terms of its limit points. $\square$

Remark 4.2.2. *The product of two spaces say X and Y denoted by $X \prod Y$ is what is gotten by pairing each element from a space X with each element from a space Y. If for example we have that $X \prod Y = \{(x, y) \mid x \in X and y \in Y\}$. It forms a Cartesian product such that each of the ordered pair is a representation of the combined elements of the two spaces.*

The coproduct or disjoint union of two spaces X and Y denoted by $X \amalg Y$ is a space that combines two elements and at the same time keeping them distinct. If *for example we have two sets x, y, then $X \amalg Y = \{(x,1) \cup (y,2) : x \in X, y \in Y\}$. Then 1 and 2 are therefore used to distinguish between elements from X and Y.*

The product and co-product are explained using universal properties. The product has a property that says that if we have another space say C with projections X and Y then it corresponds to a unique map from the space C to $X \times Y$. Similarly, the co-product has a property that any space paired with the injection map from X and Y then there exists a unique map from $X \amalg Y$ to C.

Theorem 4.2.1. *The Cantor-Bendixson rank of $X(G_n)$ is n.*

Proof. In order to show that the Cantor-Bendixson rank of $X(G_n)$ is n, Waseet in [27] argued that n is a least positive integer such that $X(G_n)^n = \Phi$. First we showed that Cantor-Bendixson rank of $X(G_n) \leq n$. From [27], an ordering is said to be a limit point if there exists infinitely many orders more mixed orders than it in $X(G_n)$. If there is an order that has all the H^i such that H^i is a mixed order then it must have an isolated order since all the H^i are already mixed. Then there isn't any collection of orders that can be limit points. This therefore implies that $X(G_n)$ can only contain orders that have atmost n-1 many of $H^1 \cdots H^n$ mixed with no orders with H^i being mixed. In a similar manner, $X(G_n)^2$ at the can only include orders that involve at most (n-2) of the elements $H^1, H^2, \cdots H^n$ in combination.

Continuing this progression, $X(G_n)$ at level (n-1) can only consist of orders that do not combine any of the elements H^i. However, at the (n-1) level, $X(G_n)$ must be finite because Gn contains only a finite number of orders in which none of the elements H^i are combined.

As a result, $X(G_n)$ at level (n) is an empty set (Φ), and the Cantor-Bendixson rank of $X(G_n) \leq n$.

Next is to demonstrate the existence of a point within $X(G_n)$ that possesses a Cantor-Bendixson rank of at least (n - 1). This discovery will lead to the conclusion that the Cantor-Bendixson of $(X(G_n)) = n$.

It's important to note that, since $X(G_n)^{(n)}$ is already known to be empty, the Cantor-Bendixson rank of any point $P \in X(G_n)$ is finite. Therefore, every point in $X(G_n)$ has a finite rank.

Consequently, if we can find a point $P \in X(G_n)$ for which Cantor-Bendixson of (P) is greater than or equal to (n - 1), then $X(G_n)^{(n-1)}$ must not be empty. However, because $X(G_n)^{(n)}$ is indeed empty, we can establish that Cantor-Bendixson of $(X(G_n)) = n$, as we intended.

We define sets $O^k for 2 \leqslant k \leqslant n \in X(G_n)$ as follows: An ordering $<\in Ok$ if and only if certain conditions are met, including the positivity of σ, λ, and $h^{i\,1}_{00}$, for all i under the ordering $<$. Additionally, it requires that the direction of each H^i is positive, with the Archimedean classes of each H^i ordered accordingly. The ordering $<$ must also satisfy certain inequalities related to $h^1_0, {}^1_0 < h^{2\ \ 1}_{u_1 0} < \cdots h^{k\ \ \ \ 1}_{u_{k-1} 0} < h^{1 1}_{1 0}$ for some values $u_1 \cdots u_{k-1}$.

Notably, all orderings in Ok involve the mixing of $H_1, H_2, \cdots H_k$. We also define $O_1 \subset$ with specific conditions, such as the positivity of σ, λ, and $hi, 10$ for all i under the ordering $<$, positive direction of each Hi, and a specific induced order of Archimedean classes .

We establish the Cantor-Bendixson rank of orderings in these sets as follows: Orderings in O_n have a CB $(\prec) = 0$; those in O_{n-1} are limit points of O_n and have CB$(\prec) \geqslant 1$; for O_{n-2}, the orderings are limit points of O_{n-1} and have CB$(\prec) \geqslant 2$, and so on.

As we continue this pattern, we find that every ordering in $O_k \forall 1 \leqslant k \leqslant n-1$ is a limit point of O_{k+1}, and it follows that CB$(\prec) \geqslant n-k$. In particular, every ordering in O_1 has a Cantor-Bendixson rank of at least $n-1$.

This leads us to the conclusion that there exists an ordering in $X(G_n)$ with a Cantor-Bendixson rank of at least $n-1$, thereby completing the proof. $\square$

Theorem 4.2.2. *[31] Every countable compact Hausdorff space is homeomorphic* *to a countable successor ordinal with the usual order topology.*

Proof. This shows that if X and Y are compact spaces of rank $\alpha + 1$ and degree d, then they are the disjoint union of finitely many compact spaces of degree 1. We can then build a homeomorphism from X to Y by induction hypothesis on the rank such that there exist a sequence that is $f_1, g_2^{-1}, f_2, g_2^{-1} \cdots$ respectively from X_1 where $X_1, X_2, X_3 \cdots$ and $Y_1, Y_2 Y_3 \cdots$ are sequences of two clopen sets and roughly $X \setminus X^\alpha$ and $Y \setminus Y^\alpha$ respectively. So X_1 is a homeomorphism to some clopen set $X_3 \subseteq X_3$ from $Y_4 \subseteq Y_4$ etc. We now show that if the spaces are metrizable then its continuous. Let X_1 be a sequence of limits x then either x has a small rank that belongs to X_j and so $f(x_i)$ has a limit of $f(x)$ that is brought up by the continuity of f_j and g_j. We can therefore say that x_i has a maximal rank.

If b is an accumulation point of the sequence $f(x_i)$ with small rank, then b belongs to some clopen set Y_j. As a result, the compact set X_j contains infinitely many x_i, which is a contradiction. Thus $f(x_i)$ has only one accumulation point and therefore must converge to y.

The result states that two countable locally compact Hausdorff spaces X and Y of the same Cantor-Bendixson rank and degree are homeomorphic. □

Lemma 4.2.1. *[38] Let X be a Hausdorff space, X_P be a perfect hull space and X_S a scattered space. Then X_P is closed while X_S is always open.*

Proposition 4.2.2. *Let X_p and X_s be a perfect hull space and a scattered space of first countable Hausdorff space X. Then both X_p and X_s have Cantor-Bendixson rank.*

Proof. A perfect hull space has a Cantor-Bendixson rank because it entails taking all the closures of isolated points that lead to accumulation or limit points. In a perfect hull space, the Cantor-Bendixson rank is the measure how the limit points are clustered within the perfect hull space.

For a scattered space, the Cantor-Bendixson rank is gotten by repeatedly taking the derived sets which is also the set of limit points and eliminating it until we find a rank that has no limit points. For a case of scattered spaces, the Cantor-Bendixson process ends at 0. Since for scattered spaces there are no accumulation points within the subsets of spaces. This implies that the Cantor-Bendixson rank is 0 or rather Cantor-Bendixson derivative of the space is empty. This therefore implies that there are no limit points in the space. This is a property that is both shared with scattered spaces. Hence the scattered spaces in relation to a Cantor-Bendixson rank is that the Cantor-Bendixson rank has a rank of 0 and hence indicating lack of limit points within the space. In conclusion, both the perfect hull space and scattered space have a Cantor-Bendixson rank. $\square$

Proposition 4.2.3. *A perfect hull space and a scattered space are closed-open decomposition of a countably based Hausdorff space X.*

Proof. Let X_n be a perfect hull space and let P be a compliment of X_n; $(P \setminus X_n)$. We want to show that P is open. Let $a \in P$ and let there exists a non empty set U that contains a such that the intersection between U and X_n is empty; $U \cap X_n = \emptyset$

Since X is a countably based Hausdorff space, then there exists an open set Z containing a such that Z is disjoint from U. Since X_n is closed then $Z \cap X_n$ is closed where Z is open. Therefore $Z \cap X_n$ is non empty because it contains elements from $U \cap X_n$. This therefore implies that Z is a subset of X and hence showing that P is open.

Similarly, let X_s be a scattered space and Y be a compliment of X_S;$Y \setminus X_s$. We want to show that Y is open. Let $b \in Y$. Since X_s is a scattered space, the it's clear that it has an isolated point and therefore there exists an open set D containing b that contains no any other point of X_s other than b itself. This therefore shows that D is a subset of Y implying that Y is open

So, in summary, since both a perfect hull space and a scattered space are closed-open decomposition of a Countably based Hausdorff space, then its sufficient to say that the specific characteristic of a Hausdorff space has a relation to the Cantor-Bendixson rank. $\square$

4.3 Profinite Space

A topological space Z is said to be a **profinite space** if its homeomorphic to a limit of a diagram of finite discrete space

Also a topological space is profinite space if it's compact, Hausdorff and totally disconnected

Example 4.3.1. *Let Z be a profinite space. If Z has an isolated point, it implies that each point of Z is isolated. That is; if $x \in Z$ then $a \to xa$ is a homeomorphism. We know that $\{\{x\} \mid x \in Z\}$ is an open cover of Z, implying that x is finite by compactness*

Proposition 4.3.1. *Let $\amalg_{n\in\mathbb{N}} P^n$ be a space with an infinite Cantor-Bendixson rank. Then the space is not compact if it is not a profinite space*

Proof. Assume that $\amalg_{n\in\mathbb{N}} P^n$ space is a profinite space. This space can therefore be represented as the limit of finite discrete space.

Now constructing an open cover of the profinite space then its clear that the union of the open sets in the cover automatically covers the entire space

But it does not matter the way we choose the finite numbers of open sets from the cover because the space is not entirely covered. This shows that for any finite sub-collection of open sets then there exists a point in the space that is not covered with the open set

Since there is an open cover that does not have a finite subcover, then the profinite space P^n does not hold for compactness. This therefore shows that if a space is not compact then it is not profinite □

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

This study was set up to determine and classify first countable Hausdorff spaces. To achieve this, we constructed and characterized the quotient maps and images that are Hausdorff, introduced the minimal generative bases and sub-bases of first countable Hausdorff space and determined the Cantor-Bendixson rank of a Hausdorff space.

In chapter three, the study successfully constructed quotient spaces of first countable Hausdorff spaces, focusing on the unique properties of such spaces in the context of minimal bases. Through theoretical construction, it was demonstrated that these quotient spaces preserve continuity under specific conditions related to the separation axioms. This result emphasizes that first countable Hausdorff spaces can be effectively analyzed and classified using quotient mappings, which form essential tools for understanding the continuity and separability of such spaces within a broader topological framework.

This chapter also introduced the concept of minimal generative sub-base for a topological space for the first time. It also explored how minimal generative base and minimal generative sub-base are related while revisiting the condition for a topological space to have a minimal base. It gave a condition for a first countable topological space to have a minimal generative sub-base, and a method to obtain one from a covering. The chapter also provided some criteria for minimal generative bases and minimal generative sub-bases for general topological spaces and extended classification of Hausdorff spaces from orderability to construction of quotient images, bases and ranks. Some of the main results in this chapter regarding the characterization of Hausdorff spaces by construction of quotient maps and images include proposition 3.1.1 and 3.2.1 that show the construction of quotient maps and images of topological spaces that are Hausdorff. Proposition 3.6.1, 3.6.2 and 3.6.3 determined the minimal generative bases and sub bases using the bases. The research concluded that these minimal bases not only generate all open sets within the topology but also serve as the most efficient representation for the space's structure. This foundational result underscores the importance of minimal bases in simplifying the study of Hausdorff spaces, enabling the reduction of complex structures to essential components without losing generality or continuity.

In chapter four, we determined whether the Cantor-Bendixson rank of the Hausdorff topological space over a real line $\mathbb{R}$ is first uncountable ordinal. We also showed that the specific characteristics of a Hausdorff space has a relation to the Cantor-Bendixson rank. Some of main results in this chapter are proposition 4.1.2 that shows that the perfect hull space and scattered spaces have a Cantor-Bendixson rank, proposition 4.1.3 that shows that the specific characteristics of a Hausdorff space has a relation to the Cantor-Bendixson rank, and proposition 4.2.1 that shows that if a space is not compact then it is not profinite.By exploring the Cantor-Bendixson rank, the study extended traditional classifications of Hausdorff spaces to include ordinal-based structures, highlighting the utility of the Cantor-Bendixson rank in assessing isolated and scattered points within these spaces. The rank analysis revealed that, under the continuum hypothesis, the real line $\mathbb{R}$ attains a rank at the first uncountable ordinal. This finding affirms the role of Cantor-Bendixson rank as a powerful tool in understanding the complexity of topological spaces and their subsets, providing a structured method to examine dense and scattered subsets within Hausdorff spaces.

5.2 Recommendations

The classification of Cantor-Bendixson rank is still open. This study leaves certain gaps that can be considered in the future;

i Characterization of first countable and scattered spaces in a quotient image.

ii Determine the properties of sets with higher Cantor-Bendixson rank and investigate their structural properties.

iii Application in Functional Analysis and Operator Theory

iv Further Exploration of Applications in Metric Spaces and Continuum Theory

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